QFI QF Model Solutions
Fall 2024

1. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Understand and apply Jensen's Inequality.

Sources:

Hirsa, Ali and Neftci, Salih N., 3rd Edition Ch. 6, Ch. 9

Chin, Eric, Nel, Dian and Olafsson Ch. 1

Commentary on Question:

This question tests candidates' knowledge of Ito's isometry, martingales, and Jensen's inequality. Most candidates were able to answer part of the question. However, few candidates scored high.

Solution:

(a) Calculate $E[X_t^2]$ for $t < T$.

Commentary on Question:
Most candidates did well on this part.

By Ito's isometry, we have

$$E[X_t^2] = E\left[\left(\int_0^t 1_{\{B_u>0\}} dB_u\right)^2\right] = E\left[\int_0^t 1_{\{B_u>0\}}^2 \, du\right] = E\left[\int_0^t 1_{\{B_u>0\}} \, du\right] = \int_0^t E\left[1_{\{B_u>0\}}\right] du$$

Note that

$$E\left[1_{\{B_u>0\}}\right] = P(B_u > 0) = \frac{1}{2}$$

Combining the above results, we get

$$E[X_t^2] = \int_0^t \frac{1}{2} du = \frac{1}{2} t$$

1. **Continued**

(b) Calculate $E[X_t Y_t]$ for $t < T$.

Commentary on Question:
Most candidates were able to apply the correlation property of Ito integral and get the correct answer.

Note that $1_{\{B_u>0\}}1_{\{B_u<0\}} = 0$ for $u \in (0, T)$.

By the correlation property of Ito integral, we have

$$E[X_t Y_t] = E\left[\int_0^t 1_{\{B_u>0\}} dB_u \int_0^t 1_{\{B_u<0\}} dB_u\right] = E\left[\int_0^t 1_{\{B_u>0\}}1_{\{B_u<0\}} du\right]$$
$$= E\left[\int_0^t 0du\right] = 0$$

(c)

(i) List the three properties of a martingale.

(ii) Determine whether $\{X_t Y_t : 0 \le t \le T\}$ is a martingale with respect to the filtration $\{I_t : 0 \le t \le T\}$ by verifying whether all the three properties listed in part (c)(i) hold.

Commentary on Question:
Most candidates were able to list the conditions of martingales. However, few candidates were able to prove the second and the third properties of martingales.

(i)

The three properties of a martingale $\{S_t : 0 \le t \le T\}$ are:

1. S_t is adapted to a filtration $\{I_t : 0 \le t \le T\}$
2. Unconditional forecast is finite, i.e., $E[|S_t|] < \infty$
3. $E[S_u | I_t] = S_t$ for $t < u$

(ii)

We show that $\{X_t Y_t : 0 \le t \le T\}$ is a martingale.

By the definition of X_t and Y_t, we know that the process $X_t Y_t$ is adapted to the filtration $\{I_t : 0 \le t \le T\}$.

Second, we show that $E[|X_t Y_t|]$ is finite. This can be done as follows.

$$E[|X_t Y_t|] \le E\left[\frac{X_t^2 + Y_t^2}{2}\right] = \frac{1}{2} t < \infty$$

1. Continued

Finally, we show that for $0 \leq s < t \leq T$, $E[X_t Y_t | I_s] = X_s Y_s$.
Note that

$$X_t Y_t = X_s Y_s + X_s(Y_t - Y_s) + (X_t - X_s)Y_s + (X_t - X_s)(Y_t - Y_s)$$

Hence

$$\begin{aligned} E[X_t Y_t | I_s] &= E[X_s Y_s + X_s(Y_t - Y_s) + (X_t - X_s)Y_s + (X_t - X_s)(Y_t - Y_s) | I_s] \\ &= E[X_s Y_s | I_s] + E[X_s(Y_t - Y_s) | I_s] + E[(X_t - X_s)Y_s | I_s] \\ &+ E[(X_t - X_s)(Y_t - Y_s) | I_s] \end{aligned}$$

Since $X_s Y_s$, X_s, and Y_s are known at time s, we have $E[X_s Y_s | I_s] = X_s Y_s$. In addition, X_t and Y_t are martingales as they are Ito integrals. We have

$$E[X_s(Y_t - Y_s) | I_s] = X_s E[Y_t - Y_s | I_s] = X_s E[Y_t - Y_s] = 0$$
$$E[(X_t - X_s)Y_s | I_s] = Y_s E[X_t - X_s | I_s] = 0$$

By the correlation property of Ito integrals, we have

$$E[(X_t - X_s)(Y_t - Y_s) | I_s] = E[(X_t - X_s)(Y_t - Y_s)] = E\left[\int_s^t 1_{\{B_u > 0\}} dB_u \int_s^t 1_{\{B_u < 0\}} dB_u\right]$$
$$= E\left[\int_s^t 1_{\{B_u > 0\}} 1_{\{B_u < 0\}} du\right] = E\left[\int_s^t 0 du\right] = 0$$

Combining the above results, we just proved that $E[X_t Y_t | I_s] = X_s Y_s$.

(d) Show, using Jensen's Inequality, that $E[\ln(|X_t B_t|)] \leq \ln(t)$ for $0 < t \leq T$.

Commentary on Question:
Few candidates did well in this part. Many candidates mistakenly treated ln(x) as a convex function.

Note that $-\ln(x)$ is a convex function. By Jensen's inequality, we have

$$E[-\ln|X_t|] \geq -\ln E[|X_t|],$$

which is

$$E[\ln|X_t|] \leq \ln E[|X_t|]$$

Similarly, we have

$$E[\ln|B_t|] \leq \ln E[|B_t|]$$

Hence, we have

$$E[\ln|X_t B_t|] = E[\ln|X_t|] + E[\ln|B_t|] \leq \ln E[|X_t|] + \ln E[|B_t|]$$

Since x^2 is a convex function, by Jensen's inequality, we get

$$E[|X_t|]^2 \leq \mathrm{E}[|X_t|^2] = \mathrm{E}[X_t^2] = \frac{t}{2},$$

which gives

$$E[|X_t|] \leq \frac{\sqrt{t}}{\sqrt{2}}$$

Similarly, we have

$$E[|B_t|] \leq \sqrt{E[B_t^2]} = \sqrt{t}$$

1. Continued

Combining the above results, we get

$$E[\ln|X_t B_t|] \leq \ln\frac{\sqrt{t}}{\sqrt{2}} + \ln\sqrt{t} < \ln t$$

2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1g) Understand the distinction between complete and incomplete markets.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

Sources:

1. Neftci Ch. 2, 5, 6, 8,14

2. Chin Ch. 2.1, 4.1

Commentary on Question:

Candidates did relatively well on this problem. Part (a)(ii) was the one which was missed by the majority of candidates.

Solution:

(a)

(i) Determine the range of r for which this model is arbitrage-free.

(ii) Assess whether this model is complete for the range of r in part (a)(i).

(i)

The model is arbitrage free if the following equations are satisfied simultaneously:

At $S_0=10$: $10 \times (1 + r) = 12 \times q_1 + 8 \times (1 - q_1)$

At $S_1 = 12$: $12 \times (1+ r) = 15 \times q_2 + 10 \times (1 - q_2)$

At $S_1 = 8$: $8 \times (1 + r) = 9 \times q_3 + 5 \times (1 - q_3)$

2. Continued

Solving them for q_1, q_2 and q_3 we get:

$$q_1 = \frac{1 + r - \frac{8}{10}}{\frac{12}{10} - \frac{8}{10}} = \frac{2 + 10r}{4}$$

$$q_2 = \frac{1 + r - 10/12}{15/12 - 10/12} = \frac{2 + 12r}{5}$$

$$q_3 = \frac{1 + r - 5/8}{9/8 - 5/8} = \frac{3 + 8r}{4}$$

Since each q_i must be in the (0,1) interval, replacing q_i with 0 and 1 in the above yields:

$-1/5 < r < 1/5$
$-1/6 < r < 1/4$
$- 3/8 < r < 1/8$

The intersection of the 3 intervals (-1/6, 1/8) gives the values of r for which the model is arbitrage free.

Some candidates did not substitute 0 and 1 for the q_s, others did not intersect the 3 intervals for r, these candidates receive partial credits

(ii) The model is complete when r $\in \left(-\frac{1}{6}, \frac{1}{8}\right)$, since each r in this interval produces equivalent risk-neutral measure

This part was missed by the majority of candidates.

(b) Calculate the fair price of this option when $r = 1/9$ using the risk-neutral measure.

Fair Price of this option

$=\frac{1}{(1+r)^2} E^Q[(\max(S_1, S_2) - K)^+]$

$=\left(\frac{9}{10}\right)^2 ((\max(12, 15) - 11)^+ q_1 q_2 + (\max(12, 10) - 11)^+ q_1(1 - q_2))$

$+\left(\frac{9}{10}\right)^2 ((\max(8, 9) - 11)^+ (1 - q_1) q_3 + (\max(8, 5) - 11)^+ (1 - q_1)(1 - q_3))$

2. **Continued**

$= \left(\frac{9}{10}\right)^2 \left(4q_1q_2 + 1q_1(1-q_2)\right)$

In the case that r = 1/9, we have from part (b) that

$$q_1 = \frac{2}{4} + \frac{10}{4}\left(\frac{1}{9}\right) = \frac{7}{9} \qquad q_2 = \frac{2}{5} + \frac{12}{5}\left(\frac{1}{9}\right) = \frac{2}{3}$$

Therefore, fair price of this option =

$$\left(\frac{9}{10}\right)^2 \left(4\left(\frac{7}{9}\right)\left(\frac{2}{3}\right) + 1\left(\frac{7}{9}\right)\left(\frac{1}{3}\right)\right) = \frac{189}{100}$$

Almost all candidates worked on this part. Common mistake here was using continuous compounding rather than discrete one. Some candidates did not calculate the option payoff correctly. In both cases, partial credit was given if the rest of the calculations were correct.

3. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Chin, Eric, Nel, Dian and Olafsson Ch. 5

Commentary on Question:

This question evaluates candidates' understanding of martingale properties. However, quite a few candidates mistakenly attempted to demonstrate Brownian motion instead. Most candidates managed to get part (a) correct but found it challenging to complete part (b). Almost no candidate can provide a satisfactory response for part (c).

Solution:

(a) Show that $\widehat{N_t}$ is a martingale.

Commentary on Question:

Most candidates were able to prove martingale and receive full credit.

We show the following three properties:

(1) $\widehat{\mathrm{N_t}}$ is $\mathcal{F}_t$-adapted since $\mathrm{N_t}$ is so.

$\mathrm{E}\left(\left|\widehat{\mathrm{N_t}}\right|\right) = \mathrm{E}(|\mathrm{N_t} - \lambda t|) \le \mathrm{E}(\mathrm{N_t}) + \lambda t, < \infty$

(2) since $\mathrm{N_t} \ge 0$. Hence $\mathrm{E}\left(\left|\widehat{\mathrm{N_t}}\right|\right) \le 2\lambda t < \infty$

(3) $\mathrm{E}\left(\widehat{\mathrm{N_t}}|\mathcal{F}_s\right) = \mathrm{E}\left(\widehat{\mathrm{N_t}} - \widehat{\mathrm{N_s}} + \widehat{\mathrm{N_s}}|\mathcal{F}_s\right) = \mathrm{E}\left(\widehat{\mathrm{N_t}} - \widehat{\mathrm{N_s}}|\mathcal{F}_s\right) + \widehat{\mathrm{N_s}}$

$= \mathrm{E}(\mathrm{N_t} - \mathrm{N_s}|\mathcal{F}_s) - \lambda(t - s) + \widehat{\mathrm{N_s}} = \widehat{\mathrm{N_s}}.$

(b) Show that $\widehat{N_t}^2 - \lambda t$ is a martingale.

Commentary on Question:

Most candidates were not able to prove the third property of martingale and therefore receive only partial credit .

Again, we show that:

(1) $\widehat{N_t}^2 - \lambda t$ is $\mathcal{F}_t$-adapted since $\mathrm{N_t}$ is so.

(2) $\mathrm{E}\left(\left|\widehat{\mathrm{N_t}}^2 - \lambda t\right|\right) \le \mathrm{E}\left(\widehat{\mathrm{N_t}}^2\right) + \lambda t = 2\lambda t < \infty.$

(3) $\mathrm{E}\left(\widehat{\mathrm{N_t}}^2 - \lambda t\,|\mathcal{F}_s\right) = \mathrm{E}\left(\left(\widehat{\mathrm{N_t}} - \widehat{\mathrm{N_s}}\right)^2 |\mathcal{F}_s\right) + 2\mathrm{E}\left(\left(\widehat{\mathrm{N_t}} - \widehat{\mathrm{N_s}}\right)\widehat{\mathrm{N_s}}\,|\mathcal{F}_s\right) + \mathrm{E}\left(\widehat{\mathrm{N_s}}^2|\mathcal{F}_s\right) - \lambda t$

$= \lambda(t - s) + \widehat{\mathrm{N_s}}^2 - \lambda t = \widehat{\mathrm{N_s}}^2 - \lambda s.$

3. **Continued**

(c) Determine whether $X_t = \exp(\widehat{N_t})$ is a martingale.

Commentary on Question:
Most candidates were not able to explain why X_t is not a martingale.

We prove that $X_t = \exp(\widehat{N_t})$ is not a martingale by showing that $E(X_t|\mathcal{F}_s) \neq X_s$.

$$
\begin{aligned}
E\left(exp\left(\widehat{N_t}\right)|\mathcal{F}_s\right) &= e^{-\lambda t}E(e^{N_t}|\mathcal{F}_s) = e^{-\lambda t}E(e^{N_t - N_s + N_s}|\mathcal{F}_s) \\
&= e^{N_s - \lambda t}E(e^{N_t - N_s}|\mathcal{F}_s) = e^{N_s - \lambda t + \lambda(t-s)(e-1)} \\
&\neq X_s
\end{aligned}
$$

4. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1d) Understand and apply Ito's Lemma.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., Third Edition, Second Printing 2014

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Derive the SDE for $S(t)^\alpha$, where α is a constant and $0 < \alpha < 1$.

Commentary on Question:

Candidates performed well on this section. It is a straightforward application of Ito's Lemma to a simple variation of a geometric Brownian motion.

$$\frac{\partial S^\alpha}{\partial S} = \alpha S^{\alpha-1}; \ \frac{\partial^2 S^\alpha}{\partial S^2} = \alpha(\alpha-1)S^{\alpha-2}; \ \frac{\partial S^\alpha}{\partial t} = 0$$

By Ito's Lemma:

$$dS^\alpha = \frac{\partial S^\alpha}{\partial t}dt + \frac{\partial S^\alpha}{\partial S}dS + \frac{1}{2}\frac{\partial^2 S^\alpha}{\partial S^2}(dS)^2$$

$$= 0dt + \alpha S^{\alpha-1}(rSdt + \sigma S dW_t) + \frac{1}{2}\alpha(\alpha-1)S^{\alpha-2}(\sigma^2 S^2 dt)$$

$$= \left[\alpha r + \frac{1}{2}\alpha(\alpha-1)\sigma^2\right]S^\alpha dt + \alpha\sigma S^\alpha dW_t$$

Alternative solution:

Equivalently, we know $d\ln S = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$ and $\ln S^\alpha = \alpha(\ln S) \Rightarrow d(\ln S^\alpha) = \alpha * d(\ln S) = \alpha\left(r - \frac{1}{2}\sigma^2\right)dt + \alpha\sigma dW_t$.

Therefore

$dS^\alpha = d(\exp(a \ln S)) = S^\alpha\left(\alpha\left(r - \frac{1}{2}\sigma^2\right)dt + \alpha\sigma dW_t + \frac{1}{2}\alpha^2\sigma^2 dt\right) = \left[\alpha r + \frac{1}{2}\alpha(\alpha-1)\sigma^2\right]S^\alpha dt + \alpha\sigma S^\alpha dW_t$

4. Continued

(b) Show $\Pr(S(t)^\alpha \leq e^{gt}) = \Phi\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)t+\frac{gt}{\alpha}}{\sigma\sqrt{t}}\right)$, where g is a constant.

Commentary on Question:
Candidates performed well on this section. The result comes from some algebra and the normality of the innovation term.

$$S(t)^\alpha < e^{gt} \Leftrightarrow \alpha \ln S < gt \Leftrightarrow \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t < \frac{gt}{\alpha} \Leftrightarrow W_t < \frac{\frac{gt}{\alpha} - \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma}$$

Since $W_t \sim N(0, t)$, this is equivalent to $\Phi\left(\frac{\frac{gt}{\alpha}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)$ which is the desired result.

(c) Show that the price P of the contract, under the classic point-to-point design, is equal to $e^{-rT}\left[\Pr(\ln S(T) \leq 0) + \mathbb{E}\left[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}\right]\right]$

Commentary on Question:
Candidates generally did well on this question. Some candidates incorrectly equated the expectations of the indicator functions and the probabilities.

Payoff is equivalent to $\begin{cases} 1 & \text{if } S(T)^\alpha \leq 1 \\ 0 & \text{if } S(T)^\alpha > 1 \end{cases} + S(T)^\alpha \begin{cases} 0 & \text{if } S(T)^\alpha \leq 1 \\ 1 & \text{if } S(T)^\alpha > 1 \end{cases}$

$$\begin{aligned} Price &= e^{-rT}\mathbb{E}\left[1\, \mathbb{I}_{\{\ln S(T) \leq 0\}} + S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}\right] \\ &= e^{-rT}\left[\mathbb{E}\left[\mathbb{I}_{\{\ln S(T) \leq 0\}}\right] + \mathbb{E}\left[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}\right]\right] \\ &= e^{-rT}\left[\mathbb{E}\left[\mathbb{I}_{\{\ln S(T) \leq 0\}}\right] + \mathbb{E}\left[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}\right]\right] \\ &= e^{-rT}\left[\Pr(\ln S(T) \leq 0) + \mathbb{E}\left[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}\right]\right] \end{aligned}$$

(d) Prove that

$$\mathbb{E}\left[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}\right] = e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T} \int_{\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{\alpha\sigma\sqrt{T}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

Commentary on Question:
Candidates performed below expectation. A key step was to write the Wiener term as a scaled standard normal random variable.

4. Continued

We've established in part (b) that the appropriate (lower) bound is $\frac{\frac{gT}{\alpha}-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$.

Given that $S(T)^\alpha = e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T+\,\alpha\sigma W_T}$, we apply the integral definition of Expectation,

$$\mathbb{E}\left[S(T)^\alpha \mathbb{I}_{\{\ln S(T)>\,0\}}\right] = \int_{\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T+\,\alpha\sigma W_T}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}dz$$

$$= \int_{\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T+\,\alpha\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}dz$$

$$= e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T}\int_{\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{\alpha\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}dz$$

(e) Show that

$$P = e^{-rT}\left[\Phi\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{\alpha\left(r+\frac{1}{2}(\alpha-1)\sigma^2\right)T}\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]$$

Commentary on Question:
Candidates performed well on this question. Candidates needed to demonstrate work and not merely jump to the final result.

The integral from part (d) is in the same form as the hint provided. So, we can quickly simplify this portion to:

$$e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T}\int_{\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{\alpha\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}dz$$

$$= e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)T} * e^{\frac{\alpha^2\sigma^2 T}{2}}\Phi\left(\alpha\sigma\sqrt{T}-\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)$$

$$= e^{\alpha\left(r+\frac{1}{2}(\alpha-1)\sigma^2\right)T}\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

4. **Continued**

The first element of the result is merely the term in part (b), with the guarantee set to 0.

Combining all the pieces, we find:

$$Price = e^{-rT}\left[\Phi\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{\alpha\left(r+\frac{1}{2}(\alpha-1)\sigma^2\right)T}\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]$$

(f) Derive the arbitrage-free price of the double threshold design for $T = 1$, using the results from part (e).

Commentary on Question:
Most candidates did not attempt this part. Of those who attempted this part, most were able to earn at least half credit. The work from the prior parts can be extended here almost directly for 2 of the 3 payoffs. The more complex element is the middle payoff, which requires some additional consideration due to the inclusion of two finite bounds.

For a double-barrier option, the payoff will be:

$$e^g\, \mathbb{I}_{\left\{\ln S(1)\le \frac{B_1}{\alpha_1}\right\}} + S(1)^{\alpha_1}\mathbb{I}_{\left\{\frac{B_1}{\alpha_1}<\ln S(1)\le \frac{B_2}{\alpha_2}\right\}} + S(1)^{\alpha_2}\mathbb{I}_{\left\{\ln S(1)> \frac{B_2}{\alpha_2}\right\}}$$

Taking expectations of each term yields:

(i) $\mathbb{E}\left[e^g\, \mathbb{I}_{\left\{\ln S(1)\le \frac{B_1}{\alpha_1}\right\}}\right] = e^g\Phi\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)+\frac{B_1}{\alpha_1}}{\sigma}\right) = \Phi\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)+\frac{B_1}{\alpha_1}}{\sigma}\right)$

(ii) $\mathbb{E}\left[S(1)^{\alpha_2}\mathbb{I}_{\left\{\ln S(1)> \frac{B_2}{\alpha_2}\right\}}\right] = e^{\alpha_2\left(r+\frac{1}{2}(\alpha_2-1)\sigma^2\right)}\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha_2\sigma^2\right)-\frac{B_2}{\alpha_2}}{\sigma}\right)$

(iii) $\mathbb{E}\left[S(1)^{\alpha_1}\mathbb{I}_{\left\{\frac{B_1}{\alpha_1}<\ln S(1)\le \frac{B_2}{\alpha_2}\right\}}\right] = e^{\alpha_1\left(r+\frac{1}{2}(\alpha_1-1)\sigma^2\right)}\left[\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha_1\sigma^2\right)-\frac{B_1}{\alpha_1}}{\sigma}\right) - \Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha_1\sigma^2\right)-\frac{B_2}{\alpha_2}}{\sigma}\right)\right]$

Final price is sum of the above three components multiplied by e^{-r} for one year of discounting.

5. Learning Objectives:

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and be able to apply various one-factor interest rate models and various simulation techniques including Euler-Maruyama discretization and transition density methods

(2e) Understand model selection and the appropriateness to the specific purpose

Sources:

Fixed Income Securities: Valuation, Risk and Risk Management, Veronesi, Pietro, 2010

Commentary on Question:

Partial points were awarded if candidates correctly grasped the logic and formulas. Candidates are expected to understand the dynamic replication of a fixed-income portfolio.

Solution:

(a) Describe the application of the replicating portfolio in the fixed-income context for the perspective of risk management of derivative hedge and the relative value trading on the yield curve.

Commentary on Question:

Most candidates showed a strong understanding of part a).

1. **Risk Management and Hedging – Hedging Derivative Exposure**
   Traders can effectively manage risk by using replicating portfolios. These portfolios work by creating a position that offsets the risk of another position. For instance, if a trader holds a security with a certain risk exposure, they can construct a replicating portfolio to counteract this exposure. This method is especially helpful when trading complex derivatives or structured products.
2. **Arbitrage Opportunities – Relative Value Trades on the Yield Curve**
   When making relative value trades, traders typically look for discrepancies in pricing between related securities. By creating a portfolio that mirrors the targeted security, traders can identify any mispricing and capitalize on potential arbitrage opportunities. A profit opportunity arises when the value of the portfolio differs from that of the targeted security, assuming the replicating portfolio has been correctly constructed.

5. **Continued**

(b) Describe the difference between the risk management for derivative hedge and relative value trading cases for the following aspects:

(i) The set up of a replicating portfolio.

(ii) The underlying asset positions in the replicating portfolios as the derivative in hedge and original bond in relative trading approach their maturities.

Commentary on Question:
Most candidates performed well on part b).

In risk management, a portfolio is established to determine the right amount of underlying assets and cash required to protect themselves against any potential price fluctuations in the (derivative security) being hedged. This is achieved by consistently modifying the composition of the underlying assets and cash to keep the position delta. As the maturity of the hedged security approaches, the position delta approaches 1 (underlying asset value).

In trading, the hedge ratio is established to determine the right amount of assets and cash in replicating the portfolio to track the price for the original security. This is achieved by consistently modifying the composition of the portfolio through buying or selling units of the underlying asset in accordance with the hedge ratio. As the original bond approaches maturity, the asset position approaches zero (the cash position increased predominantly).

(c)

(i) Calculate the interest rate, Bond A price, Bond B price, the optimal hedge ratio, cash needed for rebalancing and interest received for each month.

(ii) Tabulate the results in a detailed table format using the provided calculated table.

(iii) Plot a chart displaying the hedge ratio and the cash position.

Commentary on Question:
The majority of candidates did not perform well. While a few candidates applied the correct formulas, they failed to fill in the table. Even if a candidate entered incorrect numbers in one field, full credit for that next field was awarded if they followed the correct logic to determine the specific field.

5. Continued

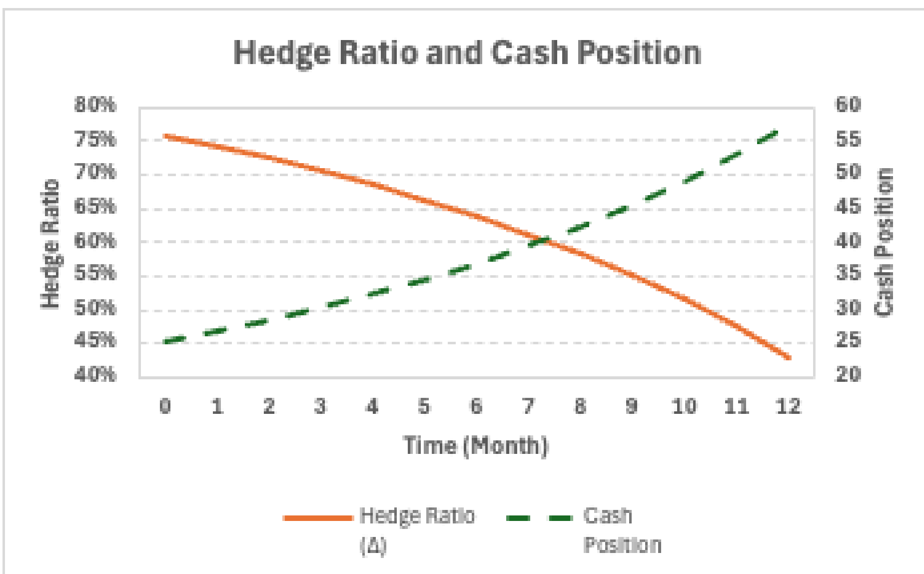
$$Z(r,t;T) = e^{A(t,T) - B(t,T)\times r}$$
$$B(t;T) = \frac{1 - e^{-\gamma^*(T-t)}}{\gamma^*},$$
$$A(t;T) = \left(B(t;T) - (T-t)\right)\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^{*2}}\right) - \frac{\sigma^2 \cdot B(t,T)^2}{4\gamma^*}$$

- Hedge ratio $\Delta_t = \frac{B(t,T_A)Z(r_t,t;T_A)}{B(t,T_B)Z(r_t,t;T_B)}$, which is subject to change based on the interest rate moves
- Cash position $C_0 = Z(r_0,0;T_A) - \Delta_0 Z(r_0,0;T_B)$
- Cash needed for rebalancing $= (\Delta_{t+dt} - \Delta_t)Z_{B,t+dt}$, where $dt = \frac{1}{12}$
- The new position in cash $C_{t+dt} = C_t + C_t r_t dt -$ Cash needed for rebalancing
- Replicating portfolio $P_t = \Delta_t Z(r_t,t;T_B) + C_t$

Time Month	r_t	T_A	T_B	$B(0,T_A)$	$B(0,T_B)$	$Z(0,T_A)$
0	5.00%	1.5000	2.5000	1.0797	1.4776	93.71
1	4.96%	1.4167	2.4167	1.0374	1.4511	94.02
2	4.92%	1.3333	2.3333	0.9935	1.4235	94.33
3	4.89%	1.2500	2.2500	0.9478	1.3948	94.65
4	4.85%	1.1667	2.1667	0.9003	1.3649	94.97
5	4.82%	1.0833	2.0833	0.8509	1.3339	95.29
6	4.79%	1.0000	2.0000	0.7996	1.3017	95.61
7	4.76%	0.9167	1.9167	0.7462	1.2682	95.94
8	4.73%	0.8333	1.8333	0.6908	1.2334	96.28
9	4.70%	0.7500	1.7500	0.6331	1.1972	96.62
10	4.67%	0.6667	1.6667	0.5732	1.1595	96.97
11	4.65%	0.5833	1.5833	0.5109	1.1204	97.32
12	4.62%	0.5000	1.5000	0.4461	1.0797	97.68

5. Continued

Time Month	$Z(0, T_B)$	Hedge Ratio (Δ)	Cash Position	Cash Needed for Rebalancing	Interest	Replicating Portfolio
0	90.19	75.93%	25.23			93.71
1	90.47	74.30%	26.81	-1.47	0.11	94.03
2	90.76	72.54%	28.52	-1.59	0.11	94.36
3	91.04	70.64%	30.36	-1.73	0.12	94.68
4	91.33	68.58%	32.37	-1.88	0.12	95.01
5	91.62	66.34%	34.55	-2.05	0.13	95.34
6	91.91	63.90%	36.94	-2.25	0.14	95.67
7	92.21	61.23%	39.55	-2.46	0.15	96.01
8	92.50	58.29%	42.42	-2.71	0.16	96.34
9	92.80	55.06%	45.59	-3.00	0.17	96.69
10	93.10	51.48%	49.10	-3.33	0.18	97.03
11	93.41	47.51%	53.00	-3.71	0.19	97.38
12	93.71	43.06%	57.37	-4.16	0.21	97.73



(d)

(i) Explain the theta-gamma relationship.

(ii) Explain its implications for dynamic hedging.

5. Continued

Commentary on Question:
Most candidates successfully identified the gamma-theta relationship. However, many candidates failed to demonstrate their understanding of its implications.

i. In any hedged portfolio – such as the ones that we obtain out of the relative value strategies – there is a tight relation between Theta (sensitivity of the portfolio to time) and Gamma (convexity of the portfolio). A security with a high Theta stands to make profits from the simple passage of time. To avoid arbitrage opportunities, such profits must be counterbalanced by a low or negative Gamma, so that the portfolio stands to lose money only because of the variation in interest rates.

The Delta-hedged portfolio Π is riskless and earns a risk-free rate, that is,
$$d\Pi = r\Pi dt$$
This portfolio must also satisfy the Fundamental Pricing Equation:
$$\frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial r} m^*(r,t) + \frac{1}{2}\frac{\partial^2 \Pi}{\partial r^2}\sigma^2 = r\Pi$$
However, since this portfolio is Delta-hedged, $\frac{\partial \Pi}{\partial \mathrm{r}} = 0$, the relation is in fact
$$\left(\frac{1}{\Pi}\frac{\partial \Pi}{\partial t}\right) + \frac{1}{2}\left(\frac{1}{\Pi}\frac{\partial^2 \Pi}{\partial r^2}\right)\sigma^2 = r$$

ii. It has a strong implication, then, namely:
High Theta $\left(\frac{1}{\Pi}\frac{\partial \Pi}{\partial t}\right) \Leftrightarrow$ Low (or even negative) convexity Gamma $\left(\frac{1}{\Pi}\frac{\partial^2 \Pi}{\partial r^2}\right)$
The intuition stems from a simple no-arbitrage argument: A positive-value portfolio with a high Theta is expected to make money because of the pure passage of time. If it was to make more money than the risk-free rate, it would be a pure arbitrage, because a trader could borrow at the risk-free rate, set up the portfolio, and wait. The negative convexity rebalances the pure arbitrage: The movement (i.e., volatility) in the interest rates tends on average to depress the portfolio value.

6. Learning Objectives:

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2g) Understand option pricing theory and techniques for interest rate derivatives

(3h) Be able to apply the models to price common interest sensitive instruments including callable bonds, bond options, caps, floors and swaptions

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 19)

Commentary on Question:

This question tests a direct application of Hull White model – the close-form solution for discount factor and its components, Black Scholes formula to calculate option on a coupon bond as a linear combination of a few options. All the formula should be listed in the Formula Sheet which was distributed and updated before the examination. Therefore, this question also tries to test candidate's familiarity to the Formula Sheet. The candidates generally did good jobs in part (a) and (b), but many of them felt intimidated by part (c).

Solution:

(a) Calculate

(i) $B(2.1; 2.5)$ and $B(2.1; 3)$

(ii) $A(2.1; 2.5)$ and $A(2.1; 3)$

Commentary on Question:

Straightforward application of Hull-White model closed-form solution formula. There is some mistakes in some of candidates' paper. Candidates should take better advantage of the Excel spreadsheet they were given during the exam to enter the formula and quickly calculate these values.

6. Continued

(i)

$$B(2.1;2.5) = \frac{1-e^{-0.309105(2.5-2.1)}}{0.309105} = 0.37626002$$

$$B(2.1;3.0) = \frac{1-e^{-0.309105(3.0-2.1)}}{0.309105} = 0.78565684$$

(ii)

$$A(2.1;2.5) = \frac{0.04}{0.310105}(1-\mathrm{e}^{-0.309105(2.5-2.1)}) - 0.04(2.5-2.1) - \frac{0.0862(2.5-2.1)^2}{2} = -0.0078456$$

$$A(2.1;3) = \frac{0.04}{0.310105}(1-\mathrm{e}^{-0.309105(3-2.1)}) - 0.04(3-2.1) - \frac{0.0862(3-2.1)^2}{2} = -0.0394847$$

(b)

(i) Calculate $Z(r_K^*, 2.1; 2.5)$

(ii) Calculate $Z(r_K^*, 2.1; 3)$

(iii) Show that $P(r_K^*, 2.1) = 95$

Commentary on Question:
Part (iii) asks for a semi-annual coupon bond price but some candidates seem not understanding that the annual coupon needs to be divided by 2 to get the semi-annual coupon. Quite a few of them got wrong results for these parts even though they correctly applied the formula.

From part (a)
$B(2.1;2.5) = 0.37626002$
$A(2.1;2.5) = -0.0078456$

$B(2.1;3) = 0.78565684$
$$A(2.1;3) = -0.0394847$$

(i)
$$Z(r_K^*, 2.1, 2.5) = e^{A(2.1,2.5)-B(2.1,2.5)r_K^*} = e^{-0.0078456-0.37626002*0.0408541} = 0.97705007$$

(ii)
$$Z(r_K^*, 2.1, 3.0) = e^{A(2.1,3.0)-B(2.1,3.0)r_K^*} = e^{-0.00394847-0.78565684*0.0408541} = 0.93091988$$

6. **Continued**

(iii)
Coupon of $1 will be paid at time 2.5 years and 3 years. Principal of $100 will be paid at time 3 years.

Thus $P(r_K^*, 2.1) = 1 * Z(r_K^*, 2.1, 2.5) + 101 * Z(r_K^*, 2.1, 3.0)$

From (i) and (ii) we have

$$P(r_K^*, 2.1) = 1 * 0.97705007 + 101 * 0.93091988 = 95$$

(c) Compute the value at t=0 of the above European Call option on the coupon bond.

Commentary on Question:
This part tests candidates' understanding of the way to calculate an option on a portfolio of cash flows which could be treated as a portfolio of option on the cash flows. This is possible because there is no correlation between cash flows (fixed coupon bond, coupon rate and the final principal repayment are all determined). It also tests candidates' familiarity on Black-Scholes formula. Most candidates did not correctly apply the formula in their calculation; a few left the entire part blank. We anticipate candidates are able to quickly and correctly apply the formula of interest rate models to many interes- rate sensitive derivatives as the LOS 2g and 2h state. Candidates need more practice to familiarize themselves to these applications.

Based on part (b), the call option on the coupon bond can be decomposed into:

(1) A call option with a strike price of 0.97705007on a bond that pays off $1 at time 2.5 years and
(2) A call option with a strike price of 94.0229074 on a bond that pays off $101 at time 3 years.

For the first option, c(1)= 0.01, principal=100

$B(2.1; 2.5) = 0.37626002$
$A(2.1; 2.5) = -0.0078456$

$B(2.1; 3) = 0.78565684$
$A(2.1; 3) = -0.0394847$

6. **Continued**

$$S_Z\,(T_o;T_1) = B(T_o;T_B) * \sqrt{\frac{\sigma^2}{2\gamma}(1-e^{-2\gamma T_O})}$$

$$S_Z\,(2.1;2.5) = 0.37626002 * \sqrt{\frac{0.2^2}{2*0.309105}(1-e^{-2*0.309105*2.1})}$$

$$S_Z\,(2.1;2.5) = 0.08160444$$

$$Z(r_0,t;T) = e^{A(t;T)-B(t;T)r_0}$$

$$Z(r_0,0;T) = e^{-r_0\mathrm{T}-\frac{0.0862T^2}{2}+\frac{r_o}{\gamma^*}(1-e^{-\gamma^*T})-\left(\frac{1-e^{-\gamma^*(T)}}{\gamma^*}\right)r_0}$$

$$Z(r_0,0;T) = e^{-\frac{0.0862T^2}{2}-r_oT}$$

$$Z(0.04,0;2.1) = e^{-\frac{0.0862*2.1^2}{2}-0.04*2.1} = 0.760278094$$

$$Z(0.04,0;2.5) = e^{-\frac{0.0862*2.5^2}{2}-0.04*2.5} = 0.691166175$$

$$d_1(1) = \frac{1}{S_Z\,(T_o;T_\mathrm{B})}ln(Z(0,r_0;T_\mathrm{B})/(K_iZ(0,r_0;T_o)\,)+\frac{S_Z\,(T_o;T_\mathrm{B})}{2}$$

$$d_1(1) = \frac{1}{0.08160444}ln\left(\frac{1*0.691166}{0.97795007*0.760278094}\right)+\frac{0.08160444}{2}$$

$$= -0.84256413$$

$d_2(1) = d_1(1) - S_Z\,(T_o;T_\mathrm{B})$ =-0.92416858

$N(d_1(1))$ =0.19973613, $N(d_2(1))$ =0.17769928

The price of the first call option
$V(r_0) = Z(0,r_0;T_B)N\big(d_1(1)\big) - K_IZ(0,r_0;T_O)N\big(d_2(1)\big)$
=0.691166175 * 0.19973613 -0.97705007 $*$ 0.760278094 $*$ 0.17769928
=0.00605054

Hence the value at time t = 0 of the European Call option on the coupon bond

= 0.00605054+ 0.010116858 x 101
= 1.02785317.

7. **Learning Objectives:**

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2f) Understand and be able to apply various model calibration techniques under both risk-neutral and real-world measures

Sources:

Calibrating Interest Rate Models, SOA Research, Oct 2023

Commentary on Question:

This question tests candidates' knowledge on the model calibration techniques.

Solution:

(a) You have one-month daily treasury bill yields (annualized) over 500 consecutive trading days in the daily_data table. There are 252 trading days per year. You would like to fit the CIR model,

$$dr = \gamma(\bar{r} - r)dt + \sqrt{\alpha r}\, dX$$

for the data set.

For this model you are considering the method based on Euler discretization and the method based on the transition density function.

Compare and contrast these two methods.

Commentary on Question:

Candidates performed below expectation on this part. Partial credits were awarded to candidates who have identified each component in the model solution.

Euler discretiztion involves discretizing the CIR SDE

$$dr_t = \gamma(\bar{r} - r_t)\, dt + \sqrt{\alpha\, r_t}\, dX_t$$

as

$$r_{t+\Delta} - r_t = \gamma(\bar{r} - r_t)\Delta + \sqrt{r\_t}\epsilon_{t+\Delta}$$

$$\epsilon_{t+\Delta} \sim N(0, \sqrt{\alpha\Delta})$$

By writing $r(i) = r_{i\Delta}, i = 0,1, \dots, n$ and

$$\alpha_1 = \gamma\bar{r}\Delta$$

$$\beta_1 = 1 - \gamma\Delta$$

7. Continued

$$\sigma = \sqrt{\alpha \Delta}$$

We can write

$$r(i) = \alpha_1 + \beta_1 r(i-1) + \sqrt{r(i-1)}\epsilon_i$$

$$\frac{r(i)}{\sqrt{r(i-1)}} = \frac{\alpha_1}{\sqrt{r(i-1)}} + \beta_1\sqrt{r(i-1)} + \epsilon_i, \qquad i = 1,2,\dots n$$

Therefore by writing

$$y_i = \frac{r(i)}{\sqrt{r(i-1)}}, \qquad x_{1i} = \frac{1}{\sqrt{r(i-1)}}, \qquad x_{2i} = \sqrt{r(i-1)}$$

The model becomes a multiple linear regression model with no intercept

$$y_i = \alpha_1 x_{1i} + \beta_1 x_{2i} + \epsilon_i$$

Maximum Likelihood Method based on Transition density

This method relies on the fact that probability density function of $r_{t+s}|r_t$ is a constant multiplier of non-central chisquared distribution. Normally we ignore the contribution from the pdf of r_0 to the likelihood function as the sample is large.

MLE is exact and should be more accurate than Euler method.
However, maximizing log likelihood function requires numerical optimization method. These are very sensitive to initial guess. The calculation of non-central chi-square in R appears to be not very stable.

(b) Calculate the estimates of $\gamma, \bar{r}$ and α based on Euler discretization.

Commentary on Question:
Candidates performed as expected on this part.

From R output

$$\alpha_1 = 0.0003346$$

$$\beta_1 = 0.9968652$$

$$\sigma = 0.01455$$

7. **Continued**

Using $\gamma = \frac{1-\beta_1}{\Delta}$

$$\gamma = 0.7899795$$

Using $\bar{r} = \frac{\alpha_1}{1-\beta_1}$

$$\bar{r} = 0.1067507$$

Using $\alpha = \frac{\sigma^2}{\Delta}$

$$\alpha = 0.0533412$$

(c) Write estimates of $\gamma, \bar{r}$ and α based on the transition density method.

Commentary on Question:
Candidates performed as expected on this part.

From the output

$$\gamma = 7.86976,$$

$$\bar{r} = 0.053306,$$

$$\alpha = 0.26746$$

(d) Recommend an estimate method between Euler discretization method and the transition density method.

Commentary on Question:
Candidates performed below expectation on this part. Partial credit was given for each component answered correctly.

From the second output even though MLE method converges, there are some warnings; warnings could be problamtics.

Two estimates are vastly different.

The diagnostics statistics for the Euler method indicates it's a good fit. However no diagnostics statistics are provided for the MLE method other than the warnings.

Based on all these considerations, the Euler estimate is recommended.

8. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(3d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(3h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016, Ch. 3, 5-7

Commentary on Question:

This question tests the candidates' understanding of the various kinds of volatilities and the interplay between the hedging assumptions vs. outcomes under a theoretical delta hedge construct. To do well on this question, the candidates need to demonstrate understanding of both the mathematical derivations of the hedging results with realized and implied volatilities, as well as the conclusions and implications under different paths of the underlying asset that could materialize.

Solution:

(a) Calculate the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt.

Commentary on Question:

Many candidates showed partial understanding of how to derive dV_t leveraging Taylor's expansion and Black-Schole Equation, though only some are able to complete all the steps. Partial marks are given in these cases.

The delta hedged portfolio has value $V_t = C_t - \Delta S_t$ at time t, where $\Delta = \frac{\partial C_t}{\partial S_t}$, thus

$$dV_t = dC_t - \Delta dS_t - rV_t dt$$

8. Continued

where the last term represents the borrowing cost of the hedge. Using Taylor's expansion of the call price:

$$dC_t = \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}dS_t^2$$

Thus,

$$dV_t = dC_t - \frac{\partial C_t}{\partial S_t}dS_t - rV_t dt = \frac{\partial C_t}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}dS_t^2 - rV_t dt$$
$$= \frac{\partial C_t}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma_R^2 S_t^2 dt - rV_t dt$$

Based on the Black-Schole Equation, value of the call C_t should satisfy the following equation with the implied volatility Σ:

$$\frac{\partial C_t}{\partial t} + rS_t\frac{\partial C_t}{\partial S_t} + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\Sigma^2 S_t^2 = rC_t$$

Thus

$$dV_t = \left(rC_t - rS_t\frac{\partial C_t}{\partial S_t} - \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\Sigma^2 S_t^2\right)dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma_R^2 S_t^2 dt - rV_t dt$$
$$= \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(\sigma_R^2 - \Sigma^2)S_t^2 dt + \left(C_t - S_t\frac{\partial C_t}{\partial S_t}\right)rdt - rV_t dt$$
$$= \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(\sigma_R^2 - \Sigma^2)S_t^2 dt$$

(b)

(i) Prove that the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt is $dV_t = e^{rt}d[e^{-rt}(C_t - C_t^R)]$

(ii) Derive the present value of the total gain or loss to maturity $\int_t^T e^{r(s-t)}dV_s$.

Commentary on Question:
Many candidates are able to partially solve the question with partial marks awarded. For part b), a minus sign is missing in the exponent (i.e. present value of total gain or loss should be $\int_t^T e^{-r(s-t)}dV_s$). Points are awarded if the candidates either followed the equation given or used the correct sign in their solutions themselves.

i) The delta hedged portfolio has value $V_t = C_t - \Delta_R S_t$ at time t, where Δ_R is computed using realized volatility.

$$dV_t = dC_t - \Delta_R dS_t - rV_t dt = dC_t - \Delta_R dS_t - r(C_t - \Delta_R S_t)dt$$
$$= dC_t - rC_t dt - \Delta_R(dS_t - rS_t dt)$$

8. Continued

If C_t is replaced by C_t^R in the above equation, then the hedged portfolio becomes risk-less, and

$$dC_t^R - rC_t^R dt - \Delta_R(dS_t - rS_t dt) = 0$$
$$dC_t^R - rC_t^R dt = \Delta_R(dS_t - rS_t dt)$$

Substituting back into the equation for dP_t, then

$$dV_t = dC_t - rC_t dt - dC_t^R + rC_t^R dt$$

Apply product rule on the right hand side of the given equation

$$e^{rt}d[e^{-rt}(C_t - C_t^R)] = e^{rt}[-re^{-rt}dt(C_t - C_t^R) + e^{-rt}d(C_t - C_t^R)]$$
$$= -r(C_t - C_t^R)dt + d(C_t - C_t^R) =] = dC_t - rC_t dt - dC_t^R + rC_t^R dt$$

Thus $dV_t = e^{rt}d[e^{-rt}(C_t - C_t^R)]$.

ii)

$$\int_t^T e^{-(s-t)r} dV_s = \int_t^T e^{-(s-t)r} e^{rs} d[e^{-rs}(C_s - C_s^R)] = \int_t^T e^{rt} d[e^{-rs}(C_s - C_s^R)]$$
$$= e^{rt}[e^{-rT}(C_T - C_T^R) - e^{-rt}(C_t - C_t^R)]$$

At maturity, value of the option is equal to the instrinsic value, i.e. $C_T = C_T^R = \max[S_T - K, 0]$, thus

$$\int_t^T e^{-(s-t)r} dV_s = C_t^R - C_t$$

(c) Compare $\int_0^{100} e^{r(s-t)} dV_s$ between the two paths if they materialize respectively, assuming

(i) The portfolio is delta-hedged based on σ_R.

(ii) The portfolio is delta-hedged based on Σ.

Commentary on Question:
Many candidates remembered the conclusions of how the hedged portfolios would behave if delta-hedged based on σ_R vs. Σ. However, many could not apply the textbook knowledge to the given construct, and misunderstood the question as that the graphs given are paths of the hedged portfolios instead of the underlying asset. Partial marks are still given for the correct knowledge from the textbook.

From part b), if the portfolio is delta-hedged based on σ_R, $\int_0^{100} e^{-(s-t)r} dV_s = C_0^R - C_0$. Values of C_0^R and C_0 are independent of the path of S_t that materializes, and thus the value is the same between the two paths.

8. Continued

From part a), if the portfolio is delta-hedged based on Σ,

$$dV_t = \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(\sigma_R^2 - \Sigma^2)S_t^2 dt$$

The infinitesimal gain or loss on the hedged portfolio is proportional to $(\sigma_R^2 - \Sigma^2)$ by the ratio of $\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}S_t^2$, which is dependent on the level of S_t, thus in this case, $\int_0^{100} e^{-(s-t)r} dV_s$ would be different between the two paths. Gamma of call options is the highest when S_t is close to the strick price, and decrease as S_t moves further into or out of money. The level of S_t is also lower for path 1. Thus $\int_0^{100} e^{-(s-t)r} dV_s$ should be lower for path 1 than path 2 in this case.

9. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3b) Identify limitations of the Black-Scholes-Merton pricing formula.

(3c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(3i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

Volatility Smile chapters 3 and 8

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014 Chapters 6 and 7

Commentary on Question:

This question was designed to test the candidate's understanding of hedging using options, the volatility smile and the limitations of the Black Schole's formula.
To get full marks candidates had to correctly calculate the value of the hedging portfolio, clearly list which assets make up the hedging portfolio, and explain two reasons a call may cost more than implied by the Black-Schole's formula.
Many candidates struggled to provide an answer to this question, but parts (a) and (b) were done well among candidates that attempted them. Most candidates were able to get partial marks for part c, but many simply listed items that can increase the cost of a call without explaining why.

Solution:

(a) Determine the value of this option strategy.

Commentary on Question:
Generally part (a) was done well by candidates that attempted it. The most common mistake was to forget the bond or not present value it.

9. Continued

The value of the option strategy will consist of a risk-free bond, and some stocks and calls. The bond needs to have a maturity of $50, this is because the Payout when the stock price is 0 is $50.

The value of the portfolio can be calculated using the following formula:

$$V(t) = Ie^{-r(T-t)} + \lambda_0 S_t + (\lambda_1 - \lambda_0)C(K_0) + (\lambda_2 - \lambda_1)C(K_1) + \cdots$$

First determine the values of lambda by dividing the change in the value of the portfolio by the change in the value of the stock at each inflection point:
Lambda_0= (75-50)/(50-0)=0.5
Lambda_1=(125-75)/(100-50)=1
Lambda_2=(0-125)/(175-100)=-5/3
Lambda_3=0/(250-175)=0

From the question we know that r=0.05, and the calls cost:

Strike (K)	Call Price(K)
0	$ 75.00
50	$ 27.93
100	$ 3.07
175	$ 0.04

Subbing these values into the formula:

$$V(t) = 50 * e^{-0.05} + 0.5 * 75 + (1 - 0.5) * 27.93 + \left(-\frac{5}{3} - 1\right) * 3.07 + \left(0 - \left(-\frac{5}{3}\right)\right) * 0.04$$

V(t)=90.91

The value of the portfolio is $90.91.

(b) Construct a replicating portfolio that fully hedges the payoff at expiration.

Commentary on Question:
Most candidates who got part (a) also got part (b). The most common mistake among those candidates was to simply calculate the value of the portfolio and never explicitly state which assets it consisted of. Purchasing ".5 units of call with strike price of 0" or ".5 units of the stock" were both accepted.

9. Continued

To replicate the payoff at expiration, purchase the following:

1) Buy a $47.56 of a risk-free bond
2) Buy 0.5 of the underlying stock or 0.5 of a call with a strike price of 0
3) Buy 0.5 of a call with a strike price of $50
4) Sell 2.67 of a call with a strike price of $100
5) Buy 1.67 of a call with a strike price of $175

(c) Provide two concrete explanations for why this happens in the market.

Commentary on Question:
To get full marks for this question a candidate needed to explain why an item would increase the cost of a call, not just list the reason. Full marks were awarded for any two valid explanations. No points were awarded for answers that applied only to puts. The most common mistake on this question was to simply list reasons why a call would cost more than Black-Scholes and not explain why.

Call options may be more expensive than what is determined using the Black-Scholes formula because of:

1) The Black-Scholes formula assumes Brownian motion and does not account for jumps in stock prices, which would be common for a micro-cap stock.
2) Out of the money options are likely to be thinly traded and therefore carry higher transaction costs which can allow for higher premiums without arbitrage opportunities.
3) Sellers may require a higher risk premium to compensate for the negative volatility convexity they will face as a result of selling out of the money options. Selling a deeply out of the money option results in a negative volatility convexity position. This means that for increased levels of volatility the sellers vega will rise, while lower levels of volatility will decrease vega. To hedge this vega the seller would need to buy volatility coverage when volatility is higher and sell volatility coverage when volatility is lower.

10. Learning Objectives:

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3e) Analyze the Greeks of common option strategies.

(3f) Appreciate how hedge strategies may go awry.

Sources:

1. QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

2. The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

Commentary on Question:

This question is to test candidates' understanding of market sensitivity Greeks and its related features as well as calculation and trading strategy associated with it.

The candidate performs relatively well in Part (a) and (c); however, most candidates did not complete the calculation for part (b). See more details below in each individual question.

Solution:

(a) Determine which Greek (Delta, Gamma, or Vega) and which expiry (1-year or 1-month) by filling the table below.

Commentary on Question:

Candidates perform well in this question. Some candidates have problems to identify Vaga and Gamma. Credits are given individually for correctly identifying Greeks and maturity with justification. However, if the corresponding Greek was not identified correctly, we do not give credits for the expiry (even if the expiry is correct).

10. Continued

	Greek	Justification to Greek
Graph A	Delta	Delta Greek is a monotonically increasing function
Graph B	Vega	Vega is bell-shaped and Vega will decrease across all ITM with shorter time to expiry.
Graph C	Gamma	Gamma is bell-shaped and gamma will increase with shorter expiry around at the money, but decrease with longer expiry at deep out-of-money and in-the-money.

	Expiry	Justification to Greek
Graph A	Line A-1: 1 month Line A-2: 1 year	Delta tends to show discontinuity on the expiration date (or very close to it)
Graph B	Line B-1: 1 year Line B-2: 1 month	Vega will decrease across all ITM with shorter time to expiry.
Graph C	Line C-1: 1 month Line C-2: 1 year	Gamma will increase with shorter expiry around at the money, but decrease with longer expiry at deep out-of-money and in-the-money.

(b) Determine, using the Black-Scholes-Merton model:

(i) the Theta if the spot price stayed the same

(ii) the Vega of the option if the stock price instantly changed to 10%

Commentary on Question:
Many candidates can identify the formula for Theta and Vega. However, many candidates have difficulties to make algebraic substitution to find the closed-form formula. Candidates are given credits for identifying the correct formula for Theta, Gamma and Rho and perform the correct substitution. Very few candidates received full credits.

There is a typo in the question (ii) – The question is asking for the Vega if the stock price instantly changed by (not "to") 10%. Candidates are not penalized if interpret it otherwise. Credits are given if correct formulas are provided.

10. Continued

(i) Theta if the spot price stayed the same

Theta(put) = $-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
Gamma = $\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$
Using algebra,
Theta = $-\frac{(N'(d_1))^2}{2*\text{Gamma}\,(T-t)} + rKe^{-r(T-t)}N(-d_2) = -\frac{(N'(d_1))^2}{2*\text{Gamma}} + rKe^{-r}N(-d_2)$

Since Rho(put) = $-K(T-t)e^{-r(T-t)}N(-d_2) = -Ke^{-r}N(-d_2)$ given (T-t) = 1

d1 = $\frac{\ln\left(\frac{S}{K}\right)+(r+0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{(5\%+0.5*0.2^2)}{0.2} = 0.35$

so $N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{{d_1}^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{0.35^2}{2}} = 0.3752$

Theta = $-\frac{\left(N'(d_1)\right)^2}{2*\text{Gamma}} - r * \text{Rho(put)}$

$$= -\frac{(0.3752)^2}{2*0.03} - 5\% * (-3) = -2.196$$

(ii) Vega of the option if the stock price instantly changed by 10%

If S dropped by 10%, then the Vega would be:

d1 = $\frac{\ln\left(\frac{S}{K}\right)+(r+0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{90}{100}\right)+(5\%+0.5*0.2^2)}{0.2} = -0.1768$

So $N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{{d_1}^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(-0.1768)^2}{2}} = 0.39275$

Vega=$SN'(d_1)\sqrt{T-t} = 90 * 0.39275 * 1 = 35.347$

(c)

(i) Describe how to construct a butterfly spread with the strike prices 80, 100, and 120.

(ii) Plot the Vega of the butterfly spread in part (c)(i) as a function of volatility of the underlying stock

10. Continued

Commentary on Question:
Many candidates are able to describe the butterfly spread strategy in (i). Some candidates give incorrect long and short positions of the call/put options.

In (ii), many candidates can plot the graph correctly given the provided formula; however, some candidates input the incorrect parameters and therefore only get partial credits.

(i) Butterfly spread can be achieved by long strangle and short straddle.

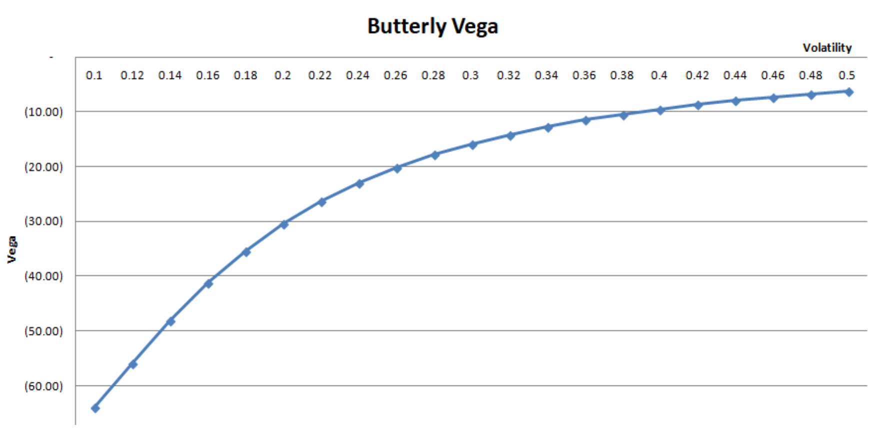
Long strangle = Long a Call (K=120) + Long a Put (K=80)
Short Straddle = Short a Call (K=100) and Short a Put (K=100)

Note that there are multiple ways to construct the butterfly position. The following alternatives are also equally acceptable to give credit:
Alternative 1: long 80 call + long 120 call + short two 100 calls.

Alternative 2: long 80 put + long 120 put + short two 100 puts.

(ii) Plot the Vega of the butterfly spread as a function of volatility

Vega of the butterfly spread as a function of the volatility is plotted below:



See spreadsheet for details.

11. **Learning Objectives:**

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(4b) Demonstrate an understanding of embedded guaranteed risk including: market, insurance, policyholder behavior, and basis risk factors.

Sources:

QFIQ-135-22: Structured Product Based Variable Annuities, Deng, Dulaney, Husson, McCann (sections 2 & 3)

Commentary on Question:

This question tests the candidate's understanding of spVA products with a cap and a buffer product design. It gets into how replicating option portfolios can be constructed for such products to support the payoff.

Solution:

(a) Show that a portfolio of a bond (with maturity value of S_0) and the following options provides the maturity payoff of this new spVA for the ranges of $S_T \geq S_0(1 + \mathrm{C})$ and $S_0 < S_T < S_0(1 + \mathrm{C})$.

- Long a floating lookback call option
- Short a European call option with strike price $S_0(1 + C)$ (i.e., out-of-money OTM) (Hint: $S_0 \geq m_0^T$)
- Short a fixed lookback put option with strike price of $S_0(1 - B)$

Commentary on Question:

Most candidates did not attempt this question. To receive full credit, the candidates needed to specify the payoffs of the floating lookback call option and the European OTM call option, then show that their combined payoffs replicate the payoff the spVA for the S_T ranges specified in the exam question.

Note that the spVA maturity payoff listed on the exam question is not replicable using the three options listed in part (a).

When using the three options listed in part (a), the correct spVA payoff should be:

$$Payoff_T^{OTM\ call} + Payoff_T^{flt\ lookback\ call} + Payoff_T^{fixed\ lookback\ put}$$

$$= \begin{cases} S_0(1+\mathrm{C}) - m_0^T + S_0, & S_T \geq S_0(1+\mathrm{C}) \text{ and } m_0^T \geq S_0(1-B) \\ S_T - m_0^T + S_0, & S_0 < S_T < S_0(1+\mathrm{C}) \text{ and } m_0^T \geq S_0(1-B) \\ S_T - m_0^T + S_0, & S_T \leq S_0 \text{ and } m_0^T \geq S_0(1-B) \\ S_0(\mathrm{C}+\mathrm{B}) + S_0, & S_T \geq S_0(1+\mathrm{C}) \text{ and } m_0^T < S_0(1-B) \\ \left(S_T - S_0(1-B)\right) + S_0, & S_0 < S_T < S_0(1+\mathrm{C}) \text{ and } m_0^T < S_0(1-B) \\ \left(S_T - S_0(1-B)\right) + S_0, & S_T \leq S_0 \text{ and } m_0^T < S_0(1-B) \end{cases}$$

11. Continued

We know that $S_T \geq m_0^T$ and $S_0 \geq m_0^T$ given that $m_0^T = \min_{0 \leq \xi \leq T} S_\xi$.

To address this issue, the grading rubric was adjusted and candidates were given full credit if they were able to show that the combined payoffs of the European OTM call and floating lookback call options replicate the spVA's payoff (as shown in the exam question) for the ranges $S_T \geq S_0(1+C)$ and $S_0 < S_T < S_0(1+C)$.

A) Short position of a European call option with strike price $S_0(1+C)$ provides the following payoff

$$Payoff_T^{OTM\ call} = \begin{cases} S_0(1+\mathrm{C}) - S_T\ , & S_T \geq S_0(1+\mathrm{C}) \\ 0 \quad , & S_T < S_0(1+\mathrm{C}) \end{cases}$$

B) Long position of a floating lookback call option provides the following payoff

$$Payoff_T^{Flt\ lookback\ call} = \begin{cases} S_T - m_0^T\ , & S_T > m_0^T \\ 0 \quad , & S_T = m_0^T \end{cases}$$

Putting A) and B) together (and given that $S_0 \geq m_0^T$) gives the following payoff:

$$Payoff_T^{OTM\ call} + Payoff_T^{flt\ lookback\ call} = \begin{cases} S_0(1+\mathrm{C}) - m_0^T \quad , & S_T \geq S_0(1+\mathrm{C}) \\ S_T - m_0^T \quad , & S_0 < S_T < S_0(1+\mathrm{C}) \end{cases}$$

Combining A), B) with a long position in bond provides the following payoff:

$$Payoff_T^{OTM\ call} + Payoff_T^{flt\ lookback\ call} + Payoff_T^{bond} = \begin{cases} S_0(1+\mathrm{C}) - m_0^T + S_0 \quad , & S_T \geq S_0(1+\mathrm{C}) \\ S_T - m_0^T + S_0 \quad , & S_0 < S_T < S_0(1+\mathrm{C}) \end{cases}$$

(b) Calculate the time-0 price of the portfolio of a bond and the options specified above.

Commentary on Question:
Many candidates did not attempt the question. To receive full credit, the candidates need to correctly calculate the price for each of the three options using Excel. Those who attempted it generally did well. Full credit was given for calculating the prices of the three options and aggregating all prices using the long/short positions specified in the question.

Answered in Excel

12. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1d) Understand and apply Ito's Lemma.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

Sources:

Nefci Ch. 10, QFIQ 134-22

Commentary on Question:

This question tests the candidates' ability to apply theories in quantitative finance to the valuation and risk management of a variable annuity with GMAB option. Specifically, a candidate needs to apply the properties of an equity return process following a Geometric Brownian Motion to derive the guarantee and cap rate under given contexts, price a GMAB option with cliquet feature, and critique on probabilistic statements based on differences in the risk neutral and real-world measures. Many candidates did not make an attempt for this question.

Solution:

(a) Show that the guarantee rate is 1.25%.

Commentary on Question:

The candidates performed below average on this section. While many candidates were able to derive the return process under the participation feature, few candidates correctly derived the expression of the price expectation.

12. Continued

From the given stock price process, $S(T)^\alpha$ also follows a Geometric Brownian Motion with the drift and volatility terms scaled by a factor of α. Therefore, we have

$$\begin{aligned}\mathbb{E}[S(T)^\alpha] &= \mathbb{E}\left[e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T+\alpha\sigma W_T}\right] \\ &= e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T}\mathbb{E}[e^{\alpha\sigma W_T}] \\ &= e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T}e^{\frac{\alpha^2\sigma^2}{2}T} \\ &= e^{\alpha\left(r+\frac{1}{2}(\alpha-1)\sigma^2\right)T}\end{aligned}$$

Substituting in the parameters, we get

$$\mathbb{E}[S(T)^\alpha] = e^{0.5\left(.04+\frac{1}{2}(0.5-1)(0.2)^2\right)T} = e^{.015T}$$

Since the guaranteed rate is 25 bps lower than the expected return under the participation factor, we have:

Guaranteed rate = 1.5%-0.25%=1.25%

(b) Derive the $\mathbb{Q}$-probability that the EIA credits the guaranteed rate in a single year.

Commentary on Question:
The candidates did poorly on this section. Among the few reasonable attempts made on this question, common mistakes include having the inequality reversed and misinterpreting the definition of the guarantee rate.

Using the results from part (a), we have:

$$\begin{aligned}\Pr(S(T)^\alpha \le e^{.0125T}) &= \Pr\left(e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T+\alpha\sigma W_T} \le e^{.0125T}\right) \\ &= \Pr\left(\alpha\left(r-\frac{\sigma^2}{2}\right)T+\alpha\sigma W_T \le .0125T\right) \\ &= \Pr\left(W_T \le \frac{.0125T-\alpha\left(r-\frac{\sigma^2}{2}\right)T}{\alpha\sigma}\right) = \Phi\left(\frac{.0125T-\alpha\left(r-\frac{\sigma^2}{2}\right)T}{\alpha\sigma\sqrt{T}}\right) \\ &= \Phi\left(\frac{.0125-0.5\left(.04-\frac{0.2^2}{2}\right)}{0.5(0.2)}\right) = \Phi(0.025) \approx 51\%\end{aligned}$$

12. **Continued**

(c) Derive the appropriate cap rate.

Commentary on Question:
The candidates performed poorly on this section. Few candidates correctly established the required probability expression.

The questions asks for the value of cap rate *c* such that

$$\Pr(S(T)^\alpha > e^{cT}) = 1 - \Pr(S(T)^\alpha \le e^{cT}) = 0.10$$

Similar to the steps in the solution to part (b), we have

$$\Pr(S(T)^\alpha \le e^{cT}) = 0.90 \Rightarrow \Phi\left(\frac{c - 0.5\left(.04 - \frac{0.2^2}{2}\right)}{0.5(0.2)}\right) = 0.9$$

$$\Rightarrow 10c - 0.1 = 1.28$$
$$\Rightarrow c = 13.8\%$$

(d) Calculate the risk-neutral price for a 5-year cliquet EIA.

Commentary on Question:
The candidates performed poorly on this section. Most of the candidates left it unanswered.

Let *g* denote the guarantee rate, we have

$$Price_{1-yr\ EIA} = e^{-r}\left\{\mathbb{E}\left[e^{g}\,\mathbb{I}_{\left\{\ln S(1) \le \frac{g}{\alpha}\right\}} + S(1)^\alpha\,\mathbb{I}_{\left\{\frac{g}{\alpha} < \ln S(1) \le \frac{c}{\alpha}\right\}} + e^{c}\,\mathbb{I}_{\left\{\ln S(1) > \frac{c}{\alpha}\right\}}\right]\right\}$$

From part (c) we know that:

$$\Pr(S(T)^\alpha \le e^{cT}) = \Phi(10c - 0.1) = \Phi(1.4 - 0.1) = \Phi(1.3) = 0.9032$$
$$\Rightarrow \Pr(S(T)^\alpha > e^{cT}) = 0.0968$$

From part (b) we know that:

$$\Pr(S(T)^\alpha \le e^{gT}) = 0.51$$

In addition, from the given stock price process, we know that:

$$S(1)^\alpha \sim LN\left(\alpha\left(r - \frac{\sigma^2}{2}\right), \alpha\sigma\right)$$

12. Continued

Hence:

$$\mathbb{E}\left[S(1)^{0.5}\mathbb{I}_{\{.025<\ln S(1)\leq .280\}}\right]$$
$$= e^{\alpha\left(r-\frac{\sigma^2}{2}\right)+\frac{\alpha^2\sigma^2}{2}}\left[\Phi\left(\frac{0.14-0.5\left(0.04-\frac{0.2^2}{2}\right)}{0.5(0.2)}-(0.5)(0.2)\right)\right.$$
$$\left.-\Phi\left(\frac{0.0125-0.5\left(0.04-\frac{0.2^2}{2}\right)}{0.5(0.2)}-(0.5)(0.2)\right)\right]$$

Combining all these results together, we have:

$$Price_{1-yr\ EIA} = 1.007728$$

The price of a 5-year cliquet is thus given by:

$$Price_{5-yr\ cliquet\ EIA} = \left(Price_{1-yr\ EIA}\right)^5 = (1.007728)^5 = 1.039242506$$

(e) Critique the following statement made by your analyst:

"By setting the cap rate such that the probability that $S(T)^{\alpha}$ exceeds cap rate is no more than 10% in a single year, we should expect to pay the cap rate approximately once every ten years."

Commentary on Question:
The candidates performed poorly on this section. While many candidates pointed out that the statement is incorrect, very few were able to give the proper rationale based on the different nature of real-world vs. risk neutral measures.

Disagree with the statement. The cap rate was set such that the risk-neutral probability of the single year return hitting the cap is 10%. Actual observation will abide by the real-world measure, which should be higher than 10% given an appropriate risk premium.