Robust Risk-Minimizing Hedging Strategies

ARC 2023

Patrice Gaillardetz

Emmanuel Osei Mireku and Saeb Hachem

August 2023

Department of Mathematics and Statistics
Concordia University

Table of contents

1. Introduction
2. Model Framework
3. Proposed Hedging Strategies
   - Local hedging strategies
   - Portfolio value as state variable
4. Numerical Examples
5. Conclusion

Introduction

Background

- Hedge by taking a financial position to offset losses from an existing position.
- Incomplete markets entail imperfect hedges
- Accept some level of uncertainty in favour of overall cheaper strategies.
- Consider model risk

Literature review

Robust optimization:

- Soyster [1973]: *Convex programming with set-inclusive constraints and applications to inexact linear programming*
- Ben-Tal and Nemirovski [1998]: *Robust convex optimization.*
- Bertsimas and Sim [2004]: *The price of robustness.*

Hedging:

- Karoui and Quenez [1995]: *Dynamic programming and pricing of contingent claims in an incomplete market.*

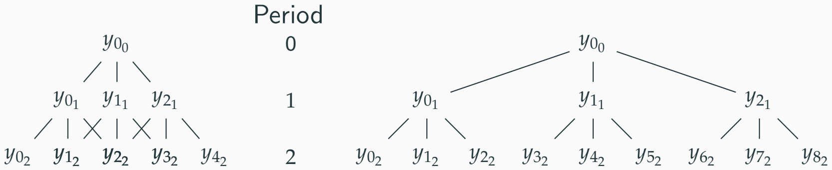
Motivation

- Model risk arises when a strategy or model is in and out of sync with the market.
- Robust optimization is an effective method for handling uncertainty.
- Robust techniques can be very expensive and the computational costs can be high.
- We propose semi-robust strategies that are model-independent, flexible, cost-effective, and computationally efficient.

Model Framework

Discrete process: Filtration

- Define Y_t is a discrete random process at the sequence of dates $t = 0, 1, 2, \ldots, T$.
- i_t is the index number for node i at period t and y_{i_t} is realization relative to node i_t.
- Denote J_t be the set of all possible states of Y_t.
- Assume a triangle filtration



Discrete process: filtration

- Assuming node i_{t-1}, Y_{i_t} has $N+1$ distinct outcomes given by $y_{i_t} u^{N-j} d^j$ for $j = 0, 1, \ldots, N$.
- Assume for volatility, σ and Δ number of trades between successive dates, $u = e^{\sigma(\Delta N)^{-0.5}}$ and $d = u^{-1}$ for recombining tree.[1]

[1] Cox et al. [1979]

Norms

- Define $\ell_{p,q}$ asymmetric norm, $\|.\|_{p,q}$ as,

$$\|Y_{i_{t-1}}\|_{p,q} = \left(\sum_{j_t \in J_t | i_{t-1}} |y^+_{j_t} - q y^-_{j_t}|^p \right)^{1/p}, \qquad (1)$$ [2]

- p is chosen to represent an investor's tolerance for risk.
- q is the measure of asymmetry.

[2] $y^+_{j_t}, y^-_{j_t} \geq 0$, $y^+_{j_t} = max(y_{j_t}, 0)$, $y^-_{j_t} = max(-y_{j_t}, 0)$ and $y_{j_t} = y^+_{j_t} - y^-_{j_t}$.

Portfolio and loss function

- $X_{i_{t-1}} = (x_{i_{t-1},k})_{k=0,1,\cdots,n}$ is a vector of either the amount or shares of assets.
- $G_{i_{t-1}}$ is a function of either the payoff vector $D_{i_{t-1}}$, or vector $C_{i_{t-1}}$, of continuation values of the hedge.
- $\mathbf{Z}_{i_{t-1}}$ is the state vector with components $\mathbf{z}_{j_t}$, $j_t \in J_t | i_{t-1}$.
- $H_{i_{t-1}}$ is a multivariate function that describes dynamic state evolution.
- $W_{i_{t-1}}(X_{i_{t-1}}, \mathbf{z}_{i_{t-1}}, \mathbf{Z}_{i_{t-1}})$ is a function that outputs a vector of accumulation of hedge portfolio at node $j_t \in J_t | i_{t-1}$ prior to claim payment.
- We define the loss random variable as

$$L_{i_{t-1}}(X_{i_{t-1}}, \mathbf{z}_{i_{t-1}}, \mathbf{Z}_{i_{t-1}}) = G_{i_{t-1}} - W_{i_{t-1}}(X_{i_{t-1}}, \mathbf{z}_{i_{t-1}}, \mathbf{Z}_{i_{t-1}})^3 \qquad (2)$$

[3] $L_{i_{t-1}}$ is convex.

Proposed Hedging Strategies

Proposed Hedging Strategies

Local hedging strategies

Super-replication (SR)

Algorithm
For all $t = T, T-1, \ldots, 1$ *and all* i_{t-1},

$$c_{i_{t-1}} = \min_{X_{i_{t-1}}} f_{i_{t-1}}(X_{i_{t-1}}) \tag{3}$$

under the constraints

$$L_{i_{t-1}}(X_{i_{t-1}}) \leq 0, \tag{4}$$

where $g_{j_T} = d_{j_T}$, $\forall j_T \in J_T | i_{T-1}$.

- If $X_{i_{t-1}}$ is the amount of each asset, $f_{i_{t-1}}(X_{i_{t-1}}) = \sum_{k=0}^{n} x_{i_{t-1},k}$.
- This is solved using dynamic programming
- The hedging strategy is given by X^* the solution of the optimization
- Hedge portfolio dominates the claim/payoff.
- The number of constraints can be reduced to those relative to the extremes.

$\ell_{p,q}$ norm as constraint (NC)

Algorithm
For all $t = T, T-1, \ldots, 1$ *and all* i_{t-1},

$$c_{i_{t-1}} = \min_{X_{i_{t-1}}} f_{i_{t-1}}(X_{i_{t-1}}) \quad (5)$$

under the constraints

$$\|L_{i_{t-1}}(X_{i_{t-1}})\|_{p,q} \leq \gamma_0, \quad (6)$$

where $g_{j_T} = d_{j_T}$, $\forall j_T \in J_T | i_{T-1}$.

- Control losses with threshold parameters γ.
- Choice of threshold depends of the desired level of conservatism and risk affinity.
- Can use state variables.

$\ell_{p,q}$ norm as objective (NO)

Algorithm
For all $t = T, T-1, \ldots, 1$ *and all* i_{t-1},

$$X^*_{i_{t-1}} = \arg\min_{X_{i_{t-1}}} \|L_{i_{t-1}}(X_{i_{t-1}})\|_{p_1,q_1} \tag{7}$$

under the constraints

$$\|L_{i_{t-1}}(X_{i_{t-1}})\|_{p_2,q_2} \leq \gamma_0, \tag{8}$$

where $g_{j_T} = d_{j_T}$, $\forall j_T \in J_T | i_{T-1}$.

- If $X^*_{i_{t-1}}$ is the optimal hedging strategy, $c_{i_{t-1}} = \sum_{k=0}^{n} x^*_{i_{t-1},k}$.
- The choice of parameter q_1 in the objective function is independent of the choice for q_2 in constraint (8).

Proposed Hedging Strategies

Portfolio value as state variable

Overview of dynamic algorithm

Algorithm
For all $t = T, T-1, \ldots, 1$ and all i_{t-1},

$$v_{i_{t-1}}(\mathbf{z}_{i_{t-1}}) = \min_{X_{i_{t-1}}, \mathbf{Z}_{i_{t-1}}} e^{-r} O(L_{i_{t-1}}(X_{i_{t-1}}, \mathbf{z}_{i_{t-1}}, \mathbf{Z}_{i_{t-1}}), V_{i_{t-1}}(\mathbf{Z}_{i_{t-1}})) \quad (9)$$

under the constraints

$$H_{i_{t-1}}(\mathbf{z}_{i_{t-1}}, \mathbf{Z}_{i_{t-1}}) = 0, \quad (10)$$

where $V_{i_{T-1}}(\mathbf{Z}_{i_{T-1}}) = 0$, and $g_{j_T} = d_{j_T}$, $\forall j_T \in J_T | i_{T-1}$.

- r is the force of interest.
- Conditional on i_{t-1}, we approximate $V_{i_{t-1}}(\mathbf{Z}_{i_{t-1}})$ by searching a polyhedron of the future cost-to-go function.
- Set $f_{i_{t-1}}(X_{i_{t-1}}) = \mathbf{z}_{i_{t-1}}$ as a state constraint.

Objective functions

- Stochastic programming:

$$\min_{X_{i_{t-1}},\mathbf{Z}_{i_{t-1}}} e^{-r}\{\|L_{i_{t-1}}(X_{i_{t-1}},\mathbf{z}_{i_{t-1}},\mathbf{Z}_{i_{t-1}})\|_{p,q}+\lambda\|V_{i_{t-1}}(\mathbf{Z}_{i_{t-1}})\|_{p,q}\} \quad (11)$$

- Dynamic coherent risk:

$$\min_{X_{i_{t-1}},\mathbf{Z}_{i_{t-1}}} e^{-r}\|L_{i_{t-1}}(X_{i_{t-1}},\mathbf{z}_{i_{t-1}},\mathbf{Z}_{i_{t-1}})+\lambda V_{i_{t-1}}(\mathbf{Z}_{i_{t-1}})\|_{p,q} \quad (12)$$

- Barrier on future risk:

$$\min_{X_{i_{t-1}},\mathbf{Z}_{i_{t-1}}} e^{-r}\|L_{i_{t-1}}(X_{i_{t-1}},\mathbf{z}_{i_{t-1}},\mathbf{Z}_{i_{t-1}})\|_{p,q}{}^{4} \quad (13)$$

[4] $V_{i_{t-1}}(\mathbf{Z}_{i_{t-1}}) \leq \gamma_0$

State variables and equations

- Portfolio value
- Self-financing strategy: portfolio value
- Transaction fees: number of risky assets.
- Other risk measures: maximum error, expected value of positive errors, CVaR, etc.

Example: Self-financing super-replication (global)

Algorithm

$$\min_{\{X_{i_0},\dots,X_{i_{T-1}}\}\in\psi} f_{i_0}(X_{i_0}) \tag{14}$$

under the constraints

$$L_{i_{T-1}}(X_{i_{T-1}}) \leq 0, \tag{15}$$

where ψ is the set of self-financing strategies.

Example: Self-financing super-replication (global)

Algorithm

For $t = T$ and all i_{T-1},

$$v_{i_{T-1}}(\mathbf{z}_{i_{T-1}}) = \min_{X_{i_{T-1}}, \mathbf{Z}_{i_{T-1}}, e_{max}} e^{-r} e_{max} \tag{16}$$

under the constraints

$$L_{i_{T-1}}(X_{i_{T-1}}, \mathbf{z}_{i_{T-1}}, \mathbf{Z}_{i_{T-1}}) \leq e_{max}, \tag{17}$$

$$f_{i_{T-1}}(X_{i_{T-1}}) = \mathbf{z}_{i_{T-1}}. \tag{18}$$

And for all $t = T-1, \ldots, 1$ and all i_{t-1},

$$v_{i_{t-1}}(\mathbf{z}_{i_{t-1}}) = \min_{X_{i_{t-1}}, \mathbf{Z}_{i_{t-1}}, \theta} e^{-r} \theta \tag{19}$$

under the constraints

$$V_{i_{t-1}}(\mathbf{Z}_{i_{t-1}}) \leq \theta, \tag{20}$$

$$f_{i_{t-1}}(X_{i_{t-1}}) = \mathbf{z}_{i_{t-1}}, \tag{21}$$

$$w_{j_t}(X_{i_{t-1}}) - \mathbf{z}_{j_t} = 0, \forall j_t \in J_t | i_{t-1}. \tag{22}$$

Need to seek the strategy such that $e_{max} = 0$ to get a super-replicating strategy.

Numerical Examples

Applications

- Consider at-the-money European call option with strike price $1 = \Upsilon_0$ and payoff at maturity $d_{j_T} = (y_{j_T} - 1)^+ \; \forall \; j_T \in J_T | i_{T-1}$.
- $g_{j_t} = \mathbf{1}_{\{t<T\}} c_{j_t} + \mathbf{1}_{\{t=T\}} d_{j_t}$.[5]
- Hedge portfolio is made up of 3 assets $x_{i_{t-1},0}, x_{i_{t-1},1}, x_{i_{t-1},2}$ (cash, stock, and a call option respectively).
- Denote the node i_t Black-Scholes price of a one-period call option with strike ky_{i_t} as $\phi_{i_t}(k)$.[6]
- For 3 assets hedging portfolio, the loss is

$$l_{j_t}(X_{i_{t-1}}) = g_{j_t} - x_{i_{t-1},0} e^r - x_{i_{t-1},1} \frac{y_{j_t}}{y_{i_{t-1}}} - x_{i_{t-1},2} (y_{j_t} - y_{i_{t-1}})^+ / \phi_{i_{t-1}}(1).$$

- Let $T = 1$, $\Delta = 12$, $N = 20$, $r = 4\%$, $\sigma = 20\%$, $\gamma_0 = 1\%$.

[5] $\mathbf{1}_{\{.\}}$ is the indicator function.
[6] k is a scalar to set the moneyness of the option asset.

Hedging errors

- The present value of hedging mismatches is

$$h = \sum_{j_t \in \{i_1, \ldots, i_T\}} e^{-rt} l_{j_t}(X_{i_{t-1}}). \tag{23}$$

- Hedger's total cost $\mathcal{H}$ is the sum of initial value of the hedge portfolio f_0 and the collection of hedging errors h.

Need probabilities for Y to find the distribution of $\mathcal{H}$

- $p_{j_t|i_{t-1}}$ transition probability from node i_{t-1} to $j_t \in J_t|i_{t-1}$.
- Let μ be the drift and $\rho = \frac{e^{\mu/(\Delta N)} - d}{u - d}$, $\mu = 8\%$. Given i_{t-1}, the conditional probabilities $p_{j_t|i_{t-1}}$ $\forall\, j_t \in J_t|i_{t-1}$ can be expressed as,

$$p_{j|i_{t-1}} = \binom{N}{j} \rho^{N-j} (1-\rho)^j, \text{ for } j = 0, 1, \ldots, N. \tag{24}$$

- Hedging errors are computed using 100,000 simulated paths.

Hedging errors (SR)

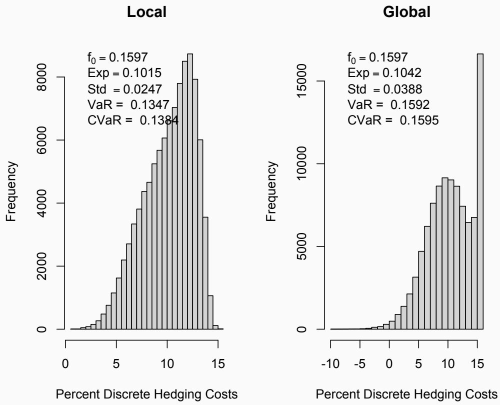


Figure 1: Distribution of simulated hedging costs for at-the-money call option with local and global super-replication strategies.

Hedging errors (NC)

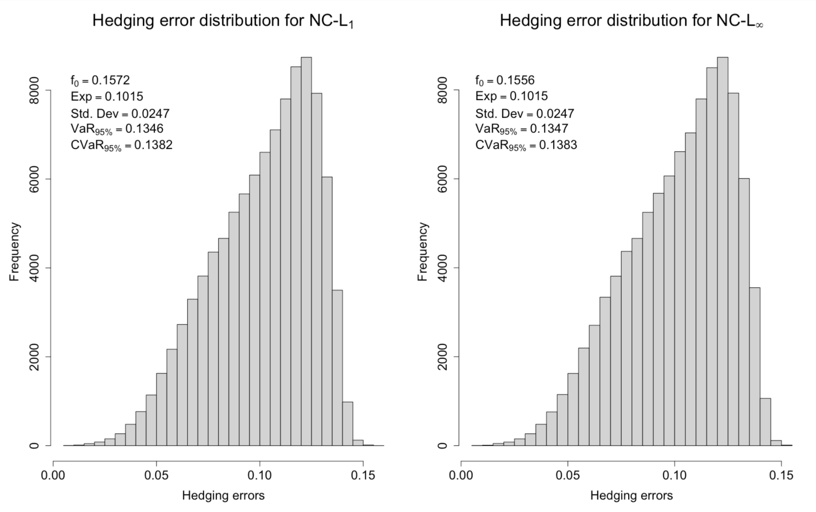


Figure 2: Distribution of simulated hedging cost for at-the-money call option with ℓ_1 (left) and ℓ_∞ (right) norms as constraint.

Hedging errors (NO)

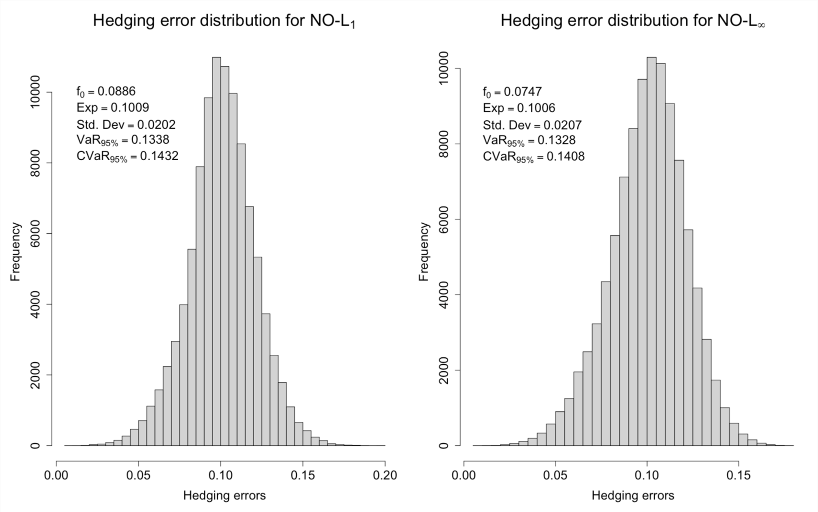


Figure 3: Distribution of simulated hedging errors for at-the-money call option with ℓ_1 (left) and ℓ_∞ (right) norms as objective under $|\sum_{j_t \in J_t | i_{t-1}} l_{j_t}(X_{i_{t-1}})| \leq 1\%$.

Number of trades & nodes

		SR		NC-$\ell_{1,0}$		NC-$\ell_{\infty,1}$		NO-$\ell_{1,1}$		NO-$\ell_{\infty,1}$	
Δ	N	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$
4	10	0.1309	0.1309	0.1257	0.1330	0.1188	0.1309	0.1000	0.1445	0.0955	0.1420
	20	0.1415	0.1415	0.1372	0.1425	0.1293	0.1415	0.1002	0.1498	0.0782	0.1499
	30	0.1467	0.1466	0.1430	0.1472	0.1346	0.1466	0.0967	0.1543	0.0626	0.1537
12	10	0.1404	0.1274	0.1379	0.1272	0.1363	0.1274	0.0970	0.1298	0.0985	0.1276
	20	0.1597	0.1383	0.1572	0.1382	0.1556	0.1383	0.0886	0.1432	0.0747	0.1408
	30	0.1715	0.1463	0.1691	0.1462	0.1674	0.1463	0.0819	0.1520	0.0477	0.1497
52	10	0.1434	0.1261	0.1428	0.1261	0.1425	0.1261	0.0946	0.1164	0.1035	0.1143
	20	0.1657	0.1365	0.1650	0.1365	0.1647	0.1365	0.0651	0.1428	0.0826	0.1223
	30	0.1804	0.1432	0.1797	0.1432	0.1794	0.1432	0.0572	0.1507	0.0403	0.1315

Table 1: Initial value f_0 and CVaR$_{95\%}$ of simulated hedging errors for at-the-money call option for SR, NC-$(\ell_{1,0}, \ell_{\infty,1})$ and NO-$(\ell_{1,1}, \ell_{\infty,1})$ strategies with $\Delta = 4, 12, 52$ number of trades and $N + 1 = 11, 21, 31$ number of nodes.

Market sensitivity

	SR		NC-$\ell_{1,0}$		NC-$\ell_{\infty,1}$		NO-$\ell_{1,1}$		NO-$\ell_{\infty,1}$	
	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$
$r = 2\%$	0.1507	0.1294	0.1482	0.1293	0.1466	0.1294	0.0776	0.1345	0.0642	0.1318
$r = 4\%$	0.1597	0.1383	0.1572	0.1382	0.1556	0.1383	0.0886	0.1432	0.0747	0.1408
$r = 6\%$	0.1689	0.1475	0.1664	0.1474	0.1649	0.1475	0.0995	0.1527	0.0857	0.1504
$\sigma = 10\%$	0.0912	0.0803	0.0867	0.0799	0.0831	0.0803	0.0569	0.0832	0.0503	0.0810
$\sigma = 20\%$	0.1597	0.1383	0.1572	0.1382	0.1556	0.1383	0.0886	0.1432	0.0747	0.1408
$\sigma = 30\%$	0.2275	0.1961	0.2258	0.1961	0.2248	0.1961	0.1196	0.2047	0.0995	0.2007

Table 2: Initial value f_0 and CVaR$_{95\%}$ of simulated hedging errors for at-the-money call option for SR, NC-$(\ell_{1,0}, \ell_{\infty,1})$ and NO-$(\ell_{1,1}, \ell_{\infty,1})$ strategies with interest rate $r = 2\%, 4\%, 6\%$, and volatility $\sigma = 10\%, 20\%, 30\%$.

Threshold & number of assets

	SR		NC-$\ell_{1,0}$		NC-$\ell_{\infty,1}$		NO-$\ell_{1,1}$		NO-$\ell_{\infty,1}$	
	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$	f_0	CVaR$_{95\%}$
$\gamma_0 = 1\%$	0.1597	0.1383	0.1572	0.1382	0.1556	0.1383	0.0886	0.1432	0.0747	0.1408
$\gamma_0 = 2\%$	0.1597	0.1383	0.1549	0.1380	0.1515	0.1383	0.0867	0.1438	0.0776	0.1401
$\gamma_0 = 3\%$	0.1597	0.1383	0.1527	0.1378	0.1475	0.1383	0.0854	0.1442	0.0803	0.1394
$\gamma_0 = \infty$	0.1597	0.1383	$-\infty$	∞	$-\infty$	∞	0.0850	0.1448	0.0781	0.1396
1-option	0.1597	0.1383	0.1572	0.1382	0.1556	0.1383	0.0886	0.1432	0.0747	0.1408
3-options	0.1464	0.1278	0.1436	0.1275	0.1423	0.1278	0.0927	0.1334	0.0969	0.1286
5-options	0.1142	0.1105	0.1122	0.1100	0.1101	0.1104	0.0982	0.1107	0.1007	0.1115

Table 3: Initial value f_0 and CVaR$_{95\%}$ of simulated hedging errors for at-the-money call option for SR, NC-$(\ell_{1,0}, \ell_{\infty,1})$ and NO-$(\ell_{1,1}, \ell_{\infty,1})$ strategies with constant threshold $\gamma_0 = 1\%, 2\%, 3\%$, and 1 call option with $k = 1$, 3 call options with $k = 0.8, 1, 1.2$, and 5 call options with $k = 0.8, 0.9, 1, 1.1, 1.2$ as assets to the hedging portfolio.

Hedging errors: NO-ℓ_2, q

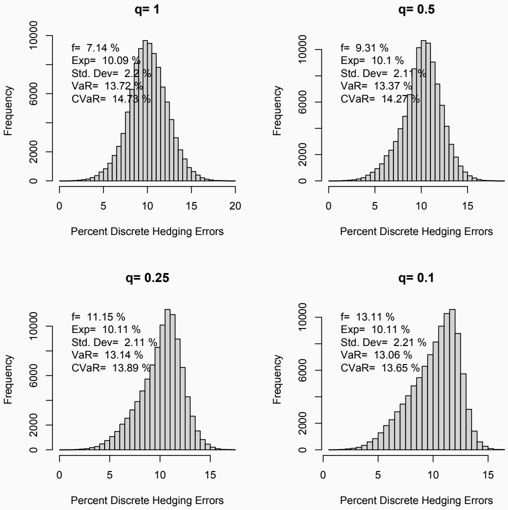


Figure 4: Distribution of simulated hedging errors for at-the-money call option with ℓ_2, q $(q = 1, 0.5, 0.25, 0.1)$ norms as objective.

Conclusion

Summary:

- Local and global semi-robust hedging strategies
- Model-independent: filtration
- Broad: different norms and asymmetric parameters
- Flexible: self-financing, transaction cost, etc.
- Cost-effective
- Easy computation: linear (Rglpk), and quadratic (Rcplex)
- Computationally efficient: curse of dimensionality (LP)

References

Aharon Ben-Tal and Arkadi Nemirovski. Robust convex optimization. *Mathematics of operations research*, 23(4):769–805, 1998.

Dimitris Bertsimas and Melvyn Sim. The price of robustness. *Operations research*, 52(1):35–53, 2004.

J.C. Cox, S.A. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of financial Economics*, 7(3):229–263, 1979.

N. El Karoui and N. Quenez. Dynamic programming and pricing of contingent claims in an incmplete market. *SIAM Journal on Control and Optimization*, 33:29–66, 1995.

Allen L Soyster. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations research*, 21 (5):1154–1157, 1973.

Thank You