STOCHASTIC CONTROL THEORY FOR OPTIMAL INVESTMENT

Maritina T. Castillo

School of Actuarial Studies

Faculty of Commerce and Economics

University of New South Wales

E-mail: tina.castillo@unsw.edu.au

and

Gilbert Parrocha

Department of Mathematics

University of the Philippines

E-mail: gilbert@math.upd.edu.ph

ABSTRACT

This paper illustrates the application of stochastic control methods in managing the risk associated with an insurance business. We present the Hamilton-Jacobi-Bellman (HJB) equation and demonstrate its use in finding the optimal investment strategy based on some given criteria. Here we consider an insurance business with a fixed amount available for investment in a portfolio consisting of one non-risky asset and one risky asset. The object of the resulting control problem is to determine the investment strategy that minimizes infinite time ruin probability. The existence of a solution to the resulting HJB equation is shown. The relevant verification lemma is not presented in this paper and a numerical algorithm is given instead. Examples applied to exponential claims are also presented.

INTRODUCTION

Business surplus is often described in ruin theory as follows:

$$\text{Surplus} = \text{Initial capital} + \text{Income} - \text{Outflow}.$$

The outflow process is determined by the claims incurred and the income process by the total premiums collected and the accrued investment income. Quantities of interest are the ruin probabilities associated with an insurance business with current surplus u:

$$\psi(u) = \Pr\{U(t) < 0 \text{ for some } t \geq 0\} \tag{1}$$

and

$$\psi(u,s) = \Pr\{U(t) < 0 \text{ for some } t \leq s\}$$

where $U(t)$ is the amount of business surplus at time t. One would like to minimize these ruin probabilities subject to the dynamics of the surplus process. In practice, for example, an actuary is often tasked to make decisions regarding new or existing insurance business. Strategies that minimize the above ruin probabilities can be considered when business policies are determined.

Recently, a lot of interest is generated by the use of mathematical tools from stochastic control theory in addressing the problem of minimizing the infinite time ruin probability defined in (1). Investment, new business, reinsurance and dividend payment are only a few of the many control variables that are adjusted dynamically in an insurance business. By means of a standard control tool such as the Hamilton-Jacobi-Bellman equation, optimal strategies can be characterized and computed, often numerically, and the smoothness of the value function can be shown. Recent applications of stochastic control tools to ruin theory include the optimization of reinsurance programs (see [6], [7], [8] and [9]), the issuance of new business (see [5]), optimal investment strategies (see [1], [3] and [4]), and simultaneous dynamic control of proportional reinsurance and investment (see [10]).

In this paper we illustrate how stochastic control tools may be applied in computing the investment strategy that would minimize the probability of ruin for a given insurance business. We consider the situation where the insurance business has current surplus u and a fixed amount a available for investment at any time $t \geq 0.$ The amount a is assumed to be independent of the business surplus. An investment portfolio consisting of one risky asset and one non-risky asset is available at each time $t \geq 0$ where a proportion $b(t) \in [0,1]$ of a is invested in the risky asset with the remaining part to be invested in the non-risky asset. This changes the dynamics of the business surplus. The investment strategy chosen is the proportion $b(t)$ of the fixed amount a to be invested in the risky asset to minimize the infinite time ruin probability. The strategy $b(t)$ is chosen predictable, i.e. it depends on all information available before time t.

DYNAMICS OF THE BUSINESS SURPLUS

We model the surplus process of an insurance business whose risk $\{R(t), t \geq 0\}$ follows a Cramér-Lundberg process with the claims process $\{S(t), t \geq 0\}$ having intensity λ and claim amount X with distribution $F(x)$ and density $f(x)$. Premiums are collected at a rate c per unit of time. Thus, the risk process is determined by

$$dR(t) = cdt - dS(t), \qquad R(0) = r.$$

The company is about to decide on an investment strategy for this business that aims to minimize the infinite time ruin probability. The investment portfolio consists of a non-risky asset whose value $B(t)$ follows

$$\frac{dB(t)}{B(t)} = \rho dt, \qquad \rho \geq 0,$$

and a risky asset whose price $Z(t)$ follows a geometric Brownian motion

$$\frac{dZ(t)}{Z(t)} = \mu dt + \sigma dW(t), \qquad \mu \geq 0, \sigma > 0.$$

The insurance business has the following investment policy:

- A fixed amount a, independent of the surplus, will be invested at any time t.
- A fraction $b(t) \in [0,1]$ of a will be invested at time t in the risky asset, the remaining part in the non-risky asset.
- The fraction $b(t)$ may change through time depending on which combination of risky and non-risky asset minimizes the infinite time ruin probability.

The investment return process $\{I(t), t \geq 0\}$ from the amount a is defined by

$$dI(t) = a[1 - b(t)]\rho dt + ab(t)\mu dt + ab(t)\sigma dW(t), \tag{2}$$

and the surplus process $\{U(t), t \geq 0\}$ of this business is seen to be

$$dU(t) = cdt - dS(t) + a[1 - b(t)]\rho dt + ab(t)\mu dt + ab(t)\sigma dW(t)$$

where $U(0) = u$. Clearly, $U(t)$ depends on the composition of the investment portfolio in which the fixed amount a is invested and is thus influenced by the investment strategy $b(t)$.

Hence, the dynamics of the surplus process can be further described as follows:

- A claim of amount X occurs with probability $\lambda dt + o(dt)$.

- No claim occurs with probability $1-\lambda dt+o(dt)$.
- An amount $cdt+o(dt)$ is received as a premium income.
- An amount $a[1-b(t)]\rho dt+o(dt)$ is received as an investment income from the non-risky asset.
- An amount $ab(t)\mu dt+ab(t)\sigma dW(t)+o(dt)$ is received as an investment income from the risky asset.

THE HAMILTON-JACOBI-BELLMAN EQUATION OF THE CONTROL PROBLEM

The control problem can be stated as follows:

$$\underset{b(t)\in[0,1]}{\text{minimize}} \quad \psi(u)=Pr\{U(t)<0 \quad \text{for some} \quad t\geq 0\}$$

subject to

$$\begin{aligned} dU(t) &= dR(t)+dI(t), \quad t\geq 0 \\ U(0) &= u. \end{aligned}$$

The solution to the control problem is found via the Hamilton-Jacobi-Bellman equation of the control problem. The solution to this equation determines the optimal proportion, $b^*(t)$.

We now define the probability of non-ruin for a business with current surplus u to be $\delta(u)=:1-\psi(u)$. We further consider two distinct cases over the time interval $[0,dt]$: that there is no claim and that there is exactly one claim during the period. If there is no claim, the surplus of the business grows to $u+cdt+dI(t)$, where $dI(t)$ is given in (2). If there is a claim, the surplus of the company reduces to $u+dI(t)-X$, where X is the random claim size. Here we assume that no premium is received during the period $[0, dt]$.

For an arbitrary strategy b at the current surplus level u, the probability of non-ruin $\delta(u)$ may now be determined. Taking expectations,

$$\delta(u) = \lambda dt E[\delta\left(u - X\right)] + (1 - \lambda dt)\delta\left(u + cdt + dI(t)\right).$$

It then follows that

$$\delta(u) = \delta(u) + \left[\frac{1}{2}\sigma^2 a^2 b^2 \delta^{''}(u) + \{c + a[1-b]\rho + ab\mu\}\delta^{'}(u) + \lambda\left[E\left[\delta\left(u - X\right) - \delta\left(u\right)\right]\right]\right]dt. \quad (3)$$

where it is seen that the proportion invested in the risky asset depends only on the current surplus level u. Here we denote the proportion simply as b. Equation (3) leads to

$$0 = \sup_b \left\{\frac{1}{2}\sigma^2 a^2 b^2 \delta^{''}(u) + \left[c + a(1-b)\rho + ab\mu\right]\delta^{'}(u) + \lambda E\left[\delta(u - X) - \delta(u)\right]\right\}, \quad (4)$$

where we have the natural conditions $\delta^{'}(u) \geq 0, \delta^{''}(u) \leq 0$ for $u > 0, \delta(u) = 0$ for $u < 0$ and $\delta(\infty) = 1$. Furthermore, we assume that $\delta(u)$ is continuous on $[0, \infty)$ and twice continuously differentiable on $(0, \infty)$. Equation (4) is called the *Hamilton-Jacobi-Bellman equation* of the control problem.

THE OPTIMAL INVESTMENT STRATEGY

Note that in (4), the quantity inside the braces is maximized by

$$\tilde{b} = \frac{(\rho - \mu)\delta^{'}(u)}{a\sigma^2 \delta^{''}(u)} \quad (5)$$

and is seen to be a function of the current surplus only. Substituting (5) in (4) results in the differential equation

$$\lambda E\left[\delta(u-X)-\delta(u)\right]=\frac{1}{2}\frac{\left(\rho-\mu\right)^2\left[\delta'(u)\right]^2}{\sigma^2\delta''(u)}-(c+a\mu)\delta'(u).$$

The solution $\delta(u)$ to this equation determines the proportion for the optimal investment strategy.

If $\tilde{b}\in[0,1]$ then it can be shown that this is the optimal proportion when the current surplus is u. However, note that the value of $\tilde{b}$ is not necessarily inside the indicated interval. For values of $\tilde{b}$ outside the interval, it will be necessary to consider the dynamics at the endpoints. Observe that if $\tilde{b}$ is less than 0, then $\rho>\mu$ and the non-risky asset is more advantageous than the risky asset. In this case, we do not invest in the risky asset. A different scenario is achieved when $\tilde{b}$ is greater than 1. This time, the risky asset is more advantageous than the non-risky asset and the optimal strategy therefore is to invest the full amount a on the risky asset. Since (4) is quadratic in b, the supremum value of the quantity inside the braces is therefore attained when $b=0$, $b=1$, or $b=\tilde{b}$. More specifically, the optimal strategy $b*$ is determined as follows:

$$b^*=\begin{cases}0 & \text{if} \quad \tilde{b}<0\\ \tilde{b} & \text{if} \quad 0\le\tilde{b}\le 1\\ 1 & \text{if} \quad \tilde{b}>1.\end{cases}$$

As a special case, if the current surplus is zero we do not invest in the risky asset. This is evident from the fact that if the current surplus is 0 and most of the amount a is invested in the risky asset then there will be greater chance of shortage of surplus to pay out possible early claims. Since the investment income can be negative, the premium income alone, accumulated for a short period of time, may not be sufficient to counter claims. It will follow from (4) that

$$0 = (c + a\rho)\delta'(0) + \lambda E\left[\delta(0 - X) - \delta(0)\right]$$
$$= (c + a\rho)\delta'(0) - \lambda\delta(0),$$

and

$$\delta'(0) = \frac{\lambda\delta(0)}{c + a\rho}. \qquad (6)$$

THE OPTIMAL NON-RUIN PROBABILITIES

We now determine the non-ruin probabilities $\delta(u)$ for cases $b^* = 0$, $b^* = 1$, and $b^* = \tilde{b}$, denoted by $\delta_0(u)$, $\delta_1(u)$, and $\delta_{\tilde{b}}(u)$, respectively.

Case 1: $b^* = 0$

The classical risk process for insurance business with premium rate $c + a\rho$ results if $b^* = 0$. Here it follows that

$$dU(t) = (c + a\rho)dt - dS(t), \qquad U(0) = u.$$

and the HJB equation simplifies to

$$0 = (c + a\rho)\delta_0'(u) + \lambda E\left[\delta_0(u - X) - \delta_0(u)\right].$$

When solved for $\delta_0'(u)$, the preceding equation is transformed to the form

$$\delta_0'(u) = \frac{\lambda}{c + a\rho} E\left[\delta_0(u) - \delta_0(u - X)\right]. \qquad (7)$$

An explicit solution for the above differential equation has been derived for an assumed distribution of X (see [2]). Generally, however, the differential equation is solved numerically.

Case 2: $b^* = 1$

A similar procedure is used to derive an expression for $\delta_1'(u)$. If b is replaced by 1, the HJB equation simplifies to

$$0 = \frac{1}{2}\sigma^2 a^2 \delta_1''(u) + (c + a\mu)\delta_1'(u) + \lambda E[\delta_1(u - X) - \delta_1(u)].$$

Equivalently,

$$\delta_1''(u) = \frac{2}{\sigma^2 a^2}\left\{\lambda E[\delta_1(u) - \delta_1(u - X)] - (c + a\mu)\delta_1'(u)\right\}. \qquad (8)$$

To simplify (8), we integrate both sides of the equation from u_1 to u, where u_1 is taken to be the least capital such that $\tilde{b} = 1$. The following differential equation results:

$$\delta_1'(u) = \frac{2}{\sigma^2 a^2}\int_{u_1}^{u}\left\{\lambda E[\delta_1(t) - \delta_1(t - X)] - (c + a\mu)\delta_1'(t)\right\}dt + \delta_1'(u_1). \qquad (9)$$

Case 3: $b^* = \tilde{b}$

Assuming that $b = \tilde{b}$ where $\tilde{b}$ is given in (5), equation (4) can be equivalently written as

$$-\frac{\delta_{\tilde{b}}''(u)}{\delta_{\tilde{b}}'(u)^2} = \frac{(\rho - \mu)^2}{2\sigma^2\left\{\lambda E[\delta_{\tilde{b}}(u) - \delta_{\tilde{b}}(u - X)] - (c + a\rho)\delta_{\tilde{b}}'(u)\right\}}. \qquad (10)$$

Integrating from 0 to u,

$$\frac{1}{\delta_{\tilde{b}}'(u)} = \frac{(\rho - \mu)^2}{2\sigma^2}\int_0^u \frac{1}{\lambda E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t - X)] - (c + a\rho)\delta_{\tilde{b}}'(t)}dt + \frac{1}{\delta_{\tilde{b}}'(0)}.$$

Solving for $\delta_{\tilde{b}}'(u)$, the following differential equation results

$$\delta_{\tilde{b}}'(u) = \left\{\frac{(\rho - \mu)^2}{2\sigma^2}\int_0^u \frac{1}{\lambda E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t - X)] - (c + a\rho)\delta_{\tilde{b}}'(t)}dt + \frac{1}{\delta_{\tilde{b}}'(0)}\right\}^{-1}. \qquad (17)$$

If $\mu > \rho$, then $\tilde{b} > 0$. As u approaches 0, $\tilde{b}$ approaches 0. Since $\delta_{\tilde{b}}'(0) > 0$ is finite, it follows that

$$\delta_{\tilde{b}}''(0^+) = -\infty.$$

The left hand side of (10) is infinite and it follows that the denominator of the right hand of the equation must be zero. Thus

$$\lambda E[\delta_{\tilde{b}}(0) - \delta_{\tilde{b}}(-X)] = (c + a\rho)\delta_{\tilde{b}}'(0^+).$$

Note that $\delta_{\tilde{b}}(-X) = 0$ and

$$\delta_{\tilde{b}}'(0^+) = \frac{\lambda \delta_{\tilde{b}}(0)}{c + a\rho}$$

which is similar to (6). Equation (4) can then be written as

$$\delta_{\tilde{b}}'(u) = \left\{ \frac{(\rho - \mu)^2}{2\sigma^2} \int_0^u \frac{1}{\lambda E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t - X)] - (c + a\rho)\delta_{\tilde{b}}'(t)} dt + \frac{c + a\rho}{\lambda \delta_{\tilde{b}}(0)} \right\}^{-1}. \qquad (11)$$

Equation (11) determines the optimal strategy b^* since once the solution to this equation is characterized, $\tilde{b}$ is solved which in turn gives the value of b^*.

EXISTENCE OF A SOLUTION TO THE HJB EQUATION

Notice that (4) determines solutions up to a multiplicative constant. That is, if $\delta(u)$ is a solution then it follows that $g(u) = \omega\delta(u)$, where $\omega > 0$ solves (4) with boundary condition $g(\infty) = \omega$. The computations below will consider a solution using $g(0) = \delta_0(0)$. A similar approach was used by C. HIPP AND M. TAKSAR [5] and H. SCHMIDLI [9].

Using the function $g(u)$ instead of $\delta(u)$, (11) can be transformed to

$$g_{\tilde{b}}^{'}(u)=\left\{\frac{(\rho-\mu)^{2}}{2\sigma^{2}}\int_{0}^{u}\frac{1}{\lambda E[g_{\tilde{b}}(t)-g_{\tilde{b}}(t-X)]-(c+a\rho)g_{\tilde{b}}^{'}(t)}dt+\frac{c+a\rho}{\lambda g_{\tilde{b}}(0)}\right\}^{-1}. \qquad (12)$$

The initial task is to ensure that the integral in (12) is finite for a surplus u. The procedure starts by showing that $g_{\tilde{b}}(u)$ exists on an interval close to zero.

First, we express $E[g(t)-g(t-X)]$ in terms of $g^{'}(t)$. By the definition of the expectation with $F(x)$ and $f(x)$, the distribution and density function of X respectively,

$$\begin{aligned}E[g(t)-g(t-X)] &= g(t)-\int_{0}^{t}g(t-x)f(x)dx\\ &= g(t)-g(0)F(t)+g(t)F(0)-\int_{0}^{t}F(x)g^{'}(t-x)dx\\ &= g(t)-g(0)F(t)-\int_{0}^{t}F(t-z)g^{'}(z)dz\\ &= g(0)[1-F(t)]+\int_{0}^{t}\left[1-F(t-z)\right]g^{'}(z)dz.\end{aligned}$$

If the expression $E[g_{\tilde{b}}-g_{\tilde{b}}(t-X)]$ in (12) is replaced by a corresponding expression based on the formula above, the equation becomes

$$g_{\tilde{b}}^{'}(u)=\left[\frac{(\rho-\mu)^{2}}{2\sigma^{2}}\int_{0}^{u}\frac{dt}{\lambda\left\{g_{\tilde{b}}(0)\left[1-F(t)\right]+\int_{0}^{t}\left[1-F(t-z)\right]g_{\tilde{b}}^{'}(z)dz\right\}-(c+a\rho)g_{\tilde{b}}^{'}(t)}+\frac{c+a\rho}{\lambda g_{\tilde{b}}(0)}\right]^{-1}.$$

Define a function $k(u)$ by

$$k(u)=\frac{\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho}-g_{\tilde{b}}^{'}(u^{2})}{u}. \qquad (13)$$

Then we have

$\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - uk(u)$

$$= \left[\frac{(\rho-\mu)^2}{2\sigma^2} \int_0^{u^2} \frac{dt}{(c+a\rho)\sqrt{t}k(\sqrt{t}) - \lambda g_{\tilde{b}}(0)F(t) + \lambda \int_0^t \left[1-F(t-z)\right]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - \sqrt{z}k(\sqrt{z})\right]dz} + \frac{c+a\rho}{\lambda g_{\tilde{b}}(0)} \right]^{-1}$$

$$= \left[\frac{(\rho-\mu)^2}{\sigma^2} \int_0^{u} \frac{dt}{(c+a\rho)k(t) - \lambda g_{\tilde{b}}(0)\frac{F(t^2)}{t} + \lambda \int_0^1 \left[1-F(t^2-t^2z)\right]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - t\sqrt{z}k(t\sqrt{z})\right]tdz} + \frac{c+a\rho}{\lambda g_{\tilde{b}}(0)} \right]^{-1}$$

Solving for $k(u)$,

$$k(u) = \frac{l(u)}{u} \frac{\lambda^2 g_{\tilde{b}}(0)^2(\rho-\mu)^2}{\lambda g_{\tilde{b}}(0)(c+a\rho)(\rho-\mu)^2 l(u) + \sigma^2(c+a\rho)^2} \qquad (14)$$

where the function $l(u)$ is defined by

$$l(u) = \int_0^u \frac{dt}{(c+a\rho)k(t) - \lambda g_{\tilde{b}}(0)\frac{F(t^2)}{t} + \lambda t \int_0^1 \left[1-F(t^2-t^2z)\right]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - t\sqrt{z}k(t\sqrt{z})\right]dz}.$$

Note that $\lim_{t\to 0}\frac{F(t^2)}{t} = \lim_{t\to 0} 2tf(t^2) = 0$. Furthermore, the functions $F(x)$ and $k(u)$ present in the integrand defining the function $l(u)$ are bounded. Therefore, the inner integral and the integrand itself are bounded and the following statements hold:

- $\lim_{u\to 0} l(u) = 0$
- $\lim_{u\to 0} \frac{l(u)}{u} = \frac{1}{(c+a\rho)\lim_{u\to 0} k(u)}$

Taking the limit of both sides of (14) as $u \to 0$ and applying the preceding properties, we have

$$\left[\lim_{u\to 0} k(u)\right]^2 = \frac{\lambda^2 g_{\tilde{b}}(0)^2 (\rho-\mu)^2}{\sigma^2 (c+a\rho)^3}.$$

Equivalently,

$$\lim_{u\to 0} k(u) = -\frac{\lambda g_{\tilde{b}}(0)(\rho-\mu)}{\sigma (c+a\rho)^{\frac{3}{2}}}.$$

Now that we know the behavior of $k(u)$ for small values of u, it is possible to generate a corresponding behavior for $g_{\tilde{b}}^{'}(u)$. Using (14),

$$g_{\tilde{b}}^{'}(u) = \frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - k(\sqrt{u})\sqrt{u}.$$

As $u \to 0$ it follows that

$$g_{\tilde{b}}^{'}(u) = \frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} + \frac{\lambda g_{\tilde{b}}(0)(\rho-\mu)}{\sigma (c+a\rho)^{\frac{3}{2}}}\sqrt{u}. \qquad (15)$$

Equation (15) gives the derivative of $g_{\tilde{b}}(u)$ for small values of u. Since $g_{\tilde{b}}(0)$ is known, the above equation can be integrated to get $g_{\tilde{b}}(u)$. Therefore, a solution to (12) exists.

A NUMERICAL ALGORITHM

The optimal strategy may be solved numerically. Given a discretization step size $\Delta u = \frac{u}{n}$, consider an initial iterate $g_{\tilde{b}}'(i\Delta u)_0$ for $i = 1, 2, ..., n$. For values of u near zero, $g_{\tilde{b}}'(i\Delta u)_0 = \delta_{\tilde{b}}'(i\Delta u)$ where $\delta_{\tilde{b}}'(i\Delta u)$ is obtained from (15). Next, define a sequence $\{g_{\tilde{b}}'(i\Delta u)_j\}$ by the recursion

$$g_{\tilde{b}}'(i\Delta u)_{j+1} = \left[\frac{(\rho-\mu)^2}{2\sigma^2} \int_0^{i\Delta u} \frac{dt}{\lambda\left\{\delta_0(0)\left[1-F(t)\right] + \int_0^t \left[1-F(t-z)\right] g_{\tilde{b}}'(z)_j\, dz\right\} - (c+a\rho) g_{\tilde{b}}'(t)_j} + \frac{c+a\rho}{\lambda\delta_0(0)} \right]^{-1}.$$

This is locally a contraction and therefore the scheme converges to $g_{\tilde{b}}'(i\Delta u)$. The initial solution $g_{\tilde{b}}'(i\Delta u)_0$ after the recursive equation is obtained by Euler scheme from (10), where $\delta_{\tilde{b}}(u)$ is replaced by $g_{\tilde{b}}(u)$. Thus,

$$g_{\tilde{b}}'(i\Delta u + \Delta u)_0 = g_{\tilde{b}}'(i\Delta u) + g_{\tilde{b}}''(i\Delta u)\Delta u.$$

The value of $\tilde{b}(i\Delta u)$ is computed using (10) and (5), where $\delta_{\tilde{b}}(u)$ is replaced by $g_{\tilde{b}}(u)$. If $\tilde{b}(i\Delta u)$ is less than 0 then the optimal proportion $b^*(i\Delta u)$ is given by $b^*(i\Delta u) = 0$. If $\tilde{b}(i\Delta u)$ is greater than 1 then $b^*(i\Delta u) = 1$, that is, $b^*(i\Delta u)$ is defined as follows.

$$b^*(i\Delta u) = \begin{cases} 0 & \text{if} \quad \tilde{b}(i\Delta u) < 0 \\ \tilde{b}(i\Delta u) & \text{if} \quad 0 \le \tilde{b}(i\Delta u) \le 1 \\ 1 & \text{if} \quad \tilde{b}(i\Delta u) > 1 \end{cases}$$

The preceding criterion determines the proportion of the amount a to be invested in the risky asset to maximize the non-ruin probability. The optimal non-ruin probability, in turn, is dependent on the above proportions.

Consider the following formulas derived from (7) and (9)

$$g_0^{'}(u) = \frac{\lambda}{c + a\rho}\left\{\delta_0(0)[1 - F(u)] + \int_0^u \left[1 - F(u-z)\right] g_0^{'}(z)dz\right\}.$$

$$g_1^{'}(u) = \frac{2}{\sigma^2 a^2}\int_0^u \left(\lambda\left\{\delta_0(0)[1 - F(t)] + \int_0^t \left[1 - F(t-z)\right] g_1^{'}(z)dz\right\} - (p + a\mu) g_1^{'}(t)\right)dt + \delta_0^{'}(0).$$

Next, define the functions $g^{'}(u)$ and $h(u)$, respectively, by

$$g^{'}(u) = \begin{cases} g_0^{'}(u) & \text{if} \quad \tilde{b}(u) < 0 \\ g_{\tilde{b}}^{'}(u) & \text{if} \quad 0 \le \tilde{b}(u) \le 1 \\ g_1^{'}(u) & \text{if} \quad \tilde{b}(u) > 1 \end{cases}$$

and

$$h(u) = \delta_0(0) + \int_0^u g^{'}(t)dt.$$

The optimal non-ruin probability $\delta(u)$ at the optimal proportion b^* is determined by the norming

$$\delta(u) = \frac{h(u)}{h(\infty)}.$$

SOME NUMERICAL EXAMPLES

In the following examples, an exponential distribution with mean γ is assumed for the claim size X. The optimal investment portfolio will be determined on a given insurance business for different investment scenarios. The examples consider two different amounts a allocated for investment and two different values of drift coefficients μ. The results obtained are very much dependent on the numerical implementation and tolerance level. The numerical implementation used a discretization size of $\Delta u = 0.001$ and a tolerance level of 1×10^{-12}.

Example 1

Consider an insurance business with premium $c = 1$ and whose aggregate claim is such that $\lambda = 1$ and $\gamma = 1$. An amount $a = 2$ will be invested in an investment portfolio with $\rho = 4\%$, $\mu = 6\%$, and $\sigma = 0.4$. Figure 1 shows the graph of the optimal proportion b^* as a function of the surplus level u.

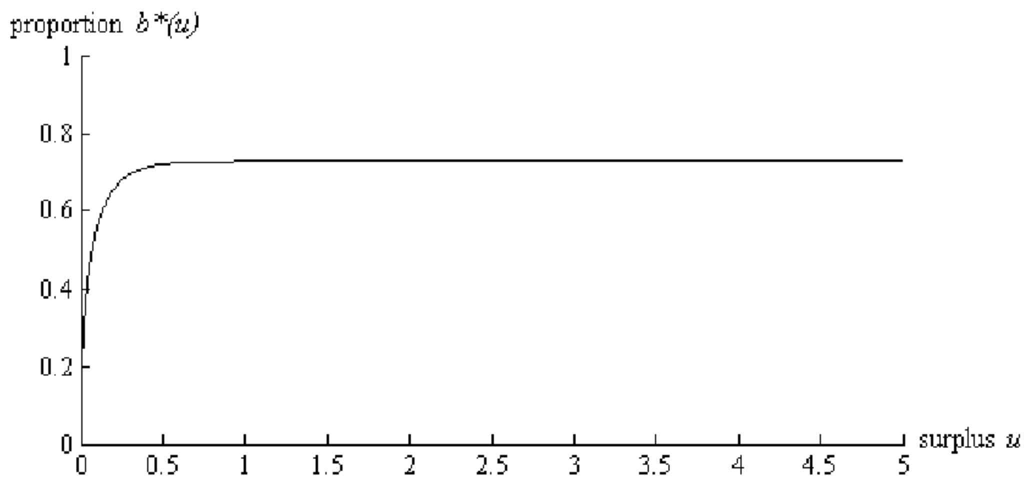

Figure 1. Graph of Example 1

The graph has an asymptote at $b^* = 0.7271$. This implies that for sufficiently large values of u, the optimal strategy is to invest a constant proportion 0.2729 on the non-risky asset and the remaining proportion 0.7271 on the risky asset. Notice that the value of b^* does not reach 1. Thus both assets comprise the optimal investment portfolio for all positive surplus levels u. Table 1 provides a summary of the optimal strategy for some small values of u. This will show how a company should invest when its surplus process is near ruin.

Surplus level u	Risky asset $b^*(u)$	Riskless asset $1 - b^*(u)$
0.0	0.0000	1.0000
0.1	0.5566	0.4434
0.2	0.6545	0.3455
0.3	0.6939	0.3061
0.4	0.7115	0.2885
0.5	0.7197	0.2803
0.6	0.7236	0.2764
0.7	0.7254	0.2746
0.8	0.7263	0.2737
0.9	0.7267	0.2733
1.0	0.7269	0.2731

Table 1. Proportion values for Example 1

It can be seen that as the surplus decreases the fraction invested on the risky asset decreases. As a company moves to a higher surplus level, the fraction of the risky asset in the investment portfolio increases. This is expected since higher surplus level implies greater capability in handling claims in the insurance business and losses incurred brought by an investment in the risky asset.

Example 2

For the same insurance business, the same amount $a = 2$ will be invested in another investment portfolio where μ is considerably higher than ρ. Here, $\rho = 4\%$, $\mu = 8\%$, and $\sigma = 0.4$. The result is different from that of Example 1. Figure 2 shows the graph of b^* for this investment portfolio.

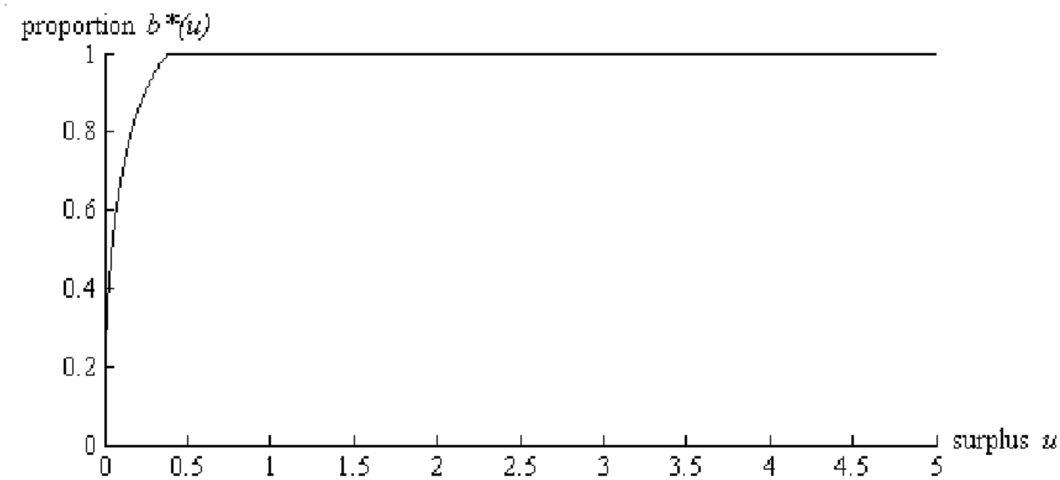

Figure 2. Graph of Example 2

Surplus level u	Risky asset $b^*(u)$	Riskless asset $1-b^*(u)$
0.0	0.0000	1.0000
0.1	0.6759	0.3241
0.2	0.8503	0.1497
0.3	0.9475	0.0525
0.4	1.0000	0.0000
0.5	1.0000	0.0000
0.6	1.0000	0.0000
0.7	1.0000	0.0000
0.8	1.0000	0.0000
0.9	1.0000	0.0000
1.0	1.0000	0.0000

Table 2. Proportion values for Example 2

It is interesting to note that $b^*(u)$ is less than 1 on the interval from 0 to 0.388. Thereafter, the optimal strategy is $b^* = 1$. At $u = 0.388$, a company has to invest fully on the risky asset to achieve the optimal non-ruin probability. At this level of surplus, the risk brought by the diffusion coefficient of the risky asset is upset by the surplus itself and the expected returns on investment. This creates a remarkable advantage of the risky asset over the non-risky asset. Table 2 gives a summary of the optimal strategy for some small values of u. Notice that the risky asset with greater drift coefficient results in a higher proportion than one with a smaller drift coefficient. This result is not surprising. Assuming that the diffusion coefficients are the same, the expected investment return will be bigger for the asset with the larger drift coefficient.

Example 3

For the same insurance business, a bigger amount $a = 5$ will be invested in the same investment scenario as in Example 1. Here, the characteristics of the investment portfolio are also $\rho = 4\%$, $\mu = 6\%$, and $\sigma = 0.4$. Observe that the result is also different from that of Example 1. Figure 3 shows the graph for b^*. Both types of asset constitute the optimal investment portfolio for all positive surplus levels since the value of b^* has not reached 1.

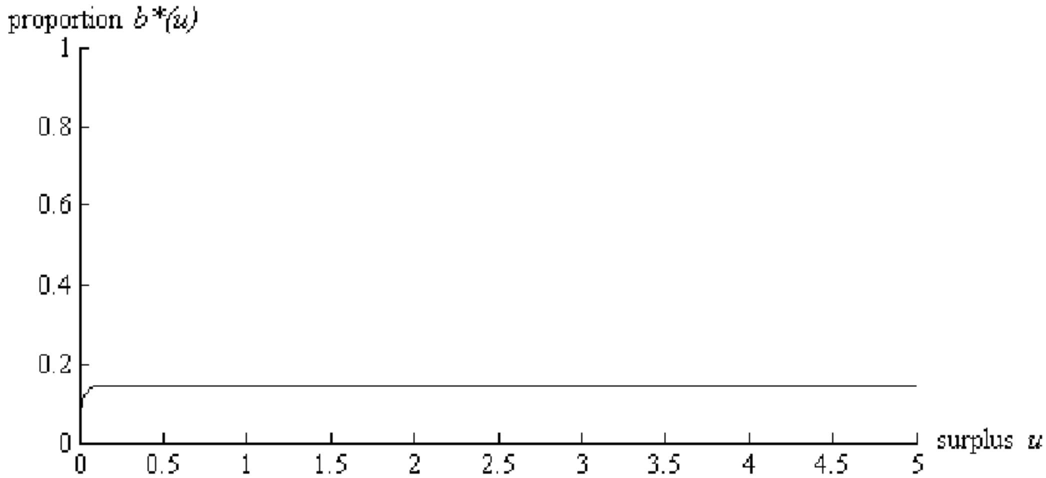

Figure 3. Graph of Example 3

The graph has an asymptote at $b^* = 0.1459$ which implies that for sufficiently large values of u, the optimal strategy is to invest a constant proportion 0.1459 on the risky asset. This is less than the corresponding proportion in Example 1. It is possible therefore to have a completely different optimal strategy with similar scenarios but having different values for a. The optimal strategy for small values of u is given in Table 3.

Surplus level u	Risky asset $b^*(u)$	Riskless asset $1-b^*(u)$
0.0	0.0000	1.0000
0.1	0.1427	0.8573
0.2	0.1457	0.8543
0.3	0.1459	0.8541
0.4	0.1459	0.8541
0.5	0.1459	0.8541
0.6	0.1459	0.8541
0.7	0.1459	0.8541
0.8	0.1459	0.8541
0.9	0.1459	0.8541
1.0	0.1459	0.8541

Table 3. Proportion values for Example 3

It is important to note here that based on the numerical results, the optimal combinations of the non-risky and the risky assets varies with the value of a allotted for investment. No single optimal combination can optimize every investment amount.

References

[1] Browne, S. (1994) *Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin.* Mathematics of Operations Research, 20, 4, 937-958.

[2] Fausett, L. (1999), *Applied Numerical Analysis Using MATLAB*, Prentice-Hall, Inc., New Jersey, 1999.

[3] Hipp, C. and Plum, M. (2003) *Optimal investment for investors with state dependent income, and for insurers.* Finance and Stochastics 7, 299-321.

[4] Hipp, C. and Plum, M. (2000) *Optimal investment for insurers.* Insurance: Mathematics and

Economics 27, 251-262.

[5] Hipp, C. and Taksar, M. (2000) *Stochastic control for optimal new business.* Insurance: Mathematics and Economics 26, 185-192.

[6] Hipp, C. and Vogt, M. (2003) *Optimal dynamic XL-reinsurance.* Preprint. Download available from www.uni-karlsruhe.de/ivw/daten/hipp/vogt.pdf

[7] Hojgaard, B. and Taksar, M. (1998) *Optimal proportional reinsurance policies for diffusion models.* Scandinavian Actuarial Journal, 81, 166-180.

[8] Hojgaard, B. and Taksar, M. (1999) *Optimal proportional reinsurance policies for diffusion models with transaction costs.* Insurance: Mathematics and Economics 22, 41-51.

[9] Schmidli, H. (1999) *Optimal proportional reinsurance policies in a dynamic setting.* Scandinavian Actuarial Journal, 77, 1-25.

[10] Schmidli, H. (2002) *On minimising the ruin probability by investment and reinsurance.* Ann. Appl. Probab. 12, 890 -907.