Pricing American Options without Expiry Date

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Abstract

This paper discusses the martingale approach for pricing American-type options without an expiry date. These options include the perpetual American put option and the perpetual maximum option in one stock case. The word “perpetual” means that the option has no expiry date. A main tool in this approach is the principle of smooth pasting.

Introduction

American options are contracts that can be exercised early, prior to the expiry date. A perpetual American option is a contract without an expiry date. It can be exercised at any time. To price an American option, it is important to determine the optimal exercise boundary (and the optimal stopping time). Myneni (1992) summarizes the essential results on the pricing of the American option. In this paper, the optimal exercise strategy related to the optimal exercise boundary will be discussed in detail in the following section.

For simplicity, I shall consider the case of one stock. For the case of one stock, I shall illustrate the pricing of perpetual American put options and the perpetual maximum options. For the optimal exercise strategy, I simplify the optimization problem as the problem of determining the optimal values of one or two parameters/threshold values. I provide a detailed derivation of these one or two endpoints of the optimal non-exercise interval.

Some classical assumptions

It is assumed that the market is frictionless. Trading is continuous. There are no taxes, no transaction cost and no restriction on borrowing or short sales. The stock price process is assumed to be a geometric Brownian motion.

Let S(t) be the price of a stock at time t and define X(t) by

$$S(t) = S(0)\, e^{X(t)}, \quad t \geq 0.$$

We assume that the process $\{X(t),\ t \geq 0\}$ is a Wiener process with instantaneous variance σ^2 and drift parameter μ.

Let r be the constant risk-free force of interest, ζ be the constant dividend yield rate of the stock. It is assumed that r and ζ are positive. Dividends of amount $\zeta S(t)dt$ are paid between time t and time t+dt. Under the risk-neutral measure, the stochastic process $\left\{e^{-rt}e^{\zeta t}S(t); t \geq 0\right\}$ is a martingale. The martingale condition is

$$E^*\left[e^{-rt}e^{\zeta t}S(t)\right] = e^{-r(0)+\zeta(0)}S(0) \tag{1}$$

or

$$E^*\left[e^{-rt+\zeta t+X(t)}\right] = e^0.$$

Thus, we have

$$(-r+\zeta+(1)\mu^*+\frac{1}{2}(1^2)\sigma^2)t = 0, \tag{2}$$

i.e.

$$\mu^* = r - \zeta - \frac{\sigma^2}{2}. \tag{3}$$

Here, the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure. Under the risk-neutral measure, $\{X(t)\}$ is a Wiener process with drift parameter μ^* which is given by (3). The diffusion parameter of $\{X(t)\}$ remains σ under the risk-neutral measure.

The optimal exercise strategy

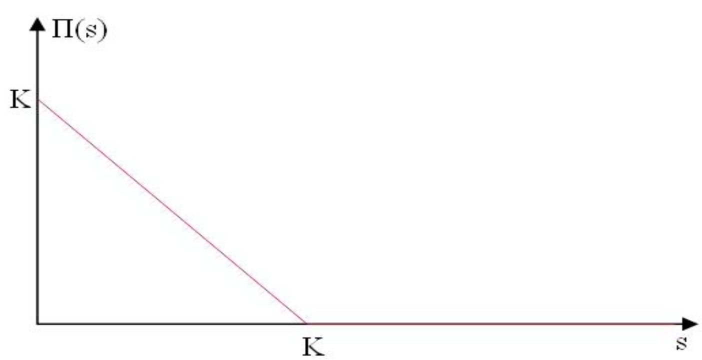
For the American option, an *optimal exercise strategy* is a *stopping time* for which the maximum value of the expected discounted payoff is attained. (The expectation is taken with respect to the risk-neutral measure.) For some perpetual options, the optimization problem can be simplified as the problem of determining the optimal values of one or two parameters.

Now let us look at a put option. If a put option with exercise price K is exercised at time t, the payoff is

$$\Pi(S(t)) = \max(K - S(t), 0),$$

where S(t) is the price of a stock at time t. See Figure 1. Note that the American put option always must be worth at least Π(S(t)) since it can be exercised at any time prior to the expiry date. This makes it more interesting and complex to evaluate than a European option which can only be exercised at the expiry date. Since an American option can be exercised at any time prior to the expiry date, choosing the optimal time to exercise is a crucial problem.

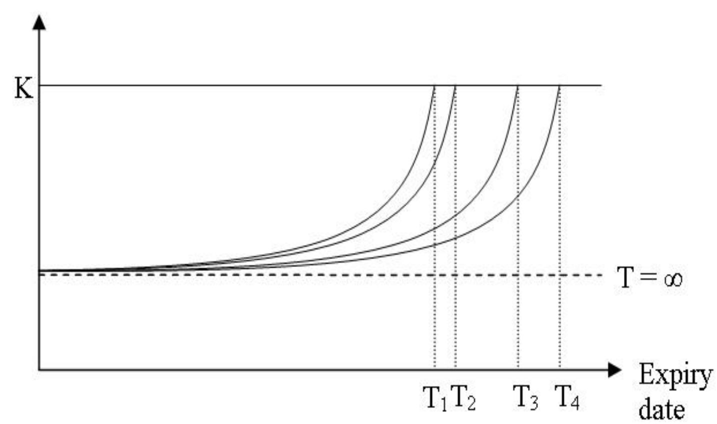
Figure 1. The payoff function of a put option



Before discussing on how to choose a stopping time, let us consider the *optimal exercise boundary* first. Once the optimal exercise boundary has been found, the price of an American put option can be obtained. The optimal exercise boundary separates the region where one should continue to hold the option and the region where one should exercise it. As shown in Figure 2, there are four optimal exercise boundaries of an

American put option corresponding to four finite expiry dates. For a specific expiry date, the corresponding optimal exercise boundary implies that one should continue to hold the option if the stock price is above the boundary curve and one should exercise the option when the embedded stock price falls on or below the optimal exercise boundary. We can see that as we extend the expiry date, the curve of the optimal exercise boundary becomes flatter and flatter. Following this trend, we can at last observe a level boundary as the expiry date tends to be infinite. Thus, we consider a level as the optimal exercise boundary for a perpetual American put option. To know more on the analysis of the optimal exercise boundary of an American put option, see Basso, Nardon and Pianca (2002), Kuske and Keller (1998) and Lindberg, Marcusson and Nordman (2002).

Figure 2. Optimal exercise boundaries of an American put option



Pricing perpetual American put options

Let us illustrate the pricing of a perpetual American put option. For an American put option with exercise price K, K < s = S(0), its payoff is

$$\Pi(S(t)) = (K - S(t))_+, \qquad (4)$$

where m_+ = Max(m, 0). If the owner of the option exercises it at a stopping time T, then he will get $(K – S(T))_+$.

As discussed in the previous section, the optimal exercise boundary of a perpetual American put option is a constant. Consider the exercise strategy that is to exercise the option as soon as the stock price falls to the level L for the first time. For 0 < L < K and L < S(0), define the stopping time T_L as

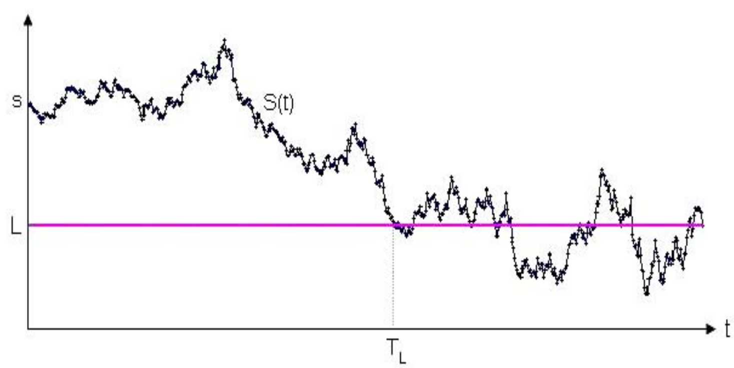
$$T_L = \min\{t \mid S(t) = L\}. \tag{5}$$

See Figure 3. The value of this exercise strategy T_L is

$$P(s; L) = E^*[e^{-rT_L}\Pi(S(T_L)) \mid S(0) = s], \tag{6}$$

where r is the risk-free force of interest.

Figure 3. The stopping time T_L



Since

$$L = S(T_L) = S(0)\,e^{X(T_L)} = s\,e^{X(T_L)} \tag{7}$$

and

$$\Pi(S(T_L)) = (K – S(T_L))_+ = (K – L)_+ = K – L,$$

formula (6) can be simplified to

$$P(s; L) = E^*[e^{-rT_L}(K - L) \mid S(0) = s]$$

$$= (K - L)E^*[e^{-rT_L} \mid S(0) = s]. \tag{8}$$

The problem is to find the value of $E^*[e^{-rT_L} \mid S(0) = s]$.

Now let us consider the stochastic process $\{e^{-rt+\beta X(t)}\}_{t\geq 0}$. This process is a martingale with respect to the risk-neutral measure if

$$E^*[e^{-rt+\beta X(t)}] = e^0 \tag{9}$$

or

$$-rt + \beta\mu^* t + \frac{1}{2}\beta^2\sigma^2 t = 0, \tag{10}$$

i.e.

$$\frac{1}{2}\sigma^2\beta^2 + \mu^*\beta - r = 0, \tag{11}$$

where μ^* is given by (3). It is obvious that there are two roots for quadratic equation (11), say β_1 and β_2. Since

$$\beta_1\beta_2 = \frac{-r}{\sigma^2/2} = \frac{-2r}{\sigma^2} < 0,$$

one root is negative and the other is positive. Assume that $\beta_1 < 0$ and $\beta_2 > 0$. For the negative root β_1, the stochastic process $\{e^{-rt+\beta_1 X(t)}\}_{0\leq t\leq T_L}$ is a bounded martingale. By the optional sampling theorem, we have

$$1 = E^*[e^{-rT_L+\beta_1 X(T_L)}] = E^*[e^{-rT_L}\left(e^{X(T_L)}\right)^{\beta_1}] = E^*[e^{-rT_L}\left(\frac{L}{s}\right)^{\beta_1}] \tag{12}$$

or

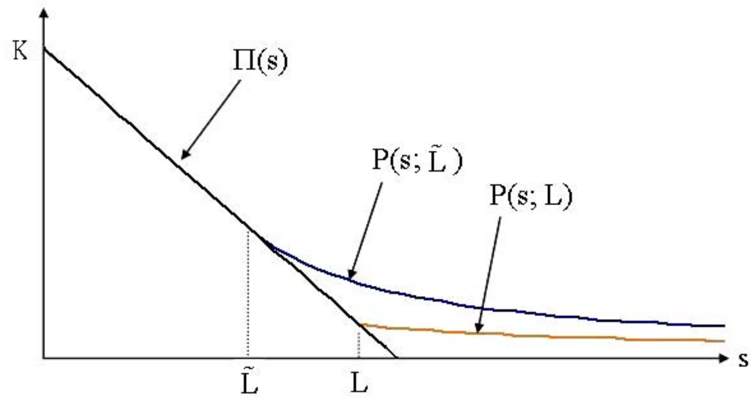
$$E^*[e^{-rT_L}] = \left(\frac{L}{s}\right)^{-\beta_1}. \tag{13}$$

Thus, for $s \geq L$ and $K > L$, it follows from (8) and (13) that the value of the strategy T_L is

$$P(s;\, L) = (K - L)\left(\frac{L}{s}\right)^{-\beta_1}, \tag{14}$$

which is represented by the red curve in Figure 4.

Figure 4. The value of the strategy T_L and The price of the perpetual American put option



Now, for the optimal exercise strategy, we seek for the value L that maximizes P(s; L). This value is called $\tilde{L}$. The optimal value $\tilde{L}$ can be obtained by differentiating P(s; L) with respect to L and setting the derivative equal to zero, i.e.,

$$\left.\frac{\partial P(s;L)}{\partial L}\right|_{L=\tilde{L}} = 0$$

or

$$-\beta_1 K\left(\frac{1}{\tilde{L}}\right)\left(\frac{\tilde{L}}{s}\right)^{-\beta_1} - (-\beta_1+1)\left(\frac{\tilde{L}}{s}\right)^{-\beta_1} = 0\,. \tag{15}$$

Solving equation (15) yields the optimal exercise boundary

$$\tilde{L} = \frac{-\beta_1}{1-\beta_1}K\,, \tag{16}$$

which is the same as (3.9) in Gerber and Shiu (1994a) if their θ_0 is β_1.

Thus, for $s \geq \tilde{L}$, it follows from (14) and (16) that the price of the perpetual American put option is

$$P(s;\tilde{L}) = (K-\tilde{L})\left(\frac{\tilde{L}}{s}\right)^{-\beta_1}$$

$$= (K - \frac{-\beta_1}{1-\beta_1}K)\left(\frac{1}{s}\frac{-\beta_1}{1-\beta_1}K\right)^{-\beta_1}$$

$$= \frac{K}{1-\beta_1}\left(\frac{-K\beta_1}{s(1-\beta_1)}\right)^{-\beta_1}, \tag{17}$$

which is represented by the blue curve in Figure 4. Note that for L < K, as L tends to $\tilde{L}$, the value of P(s; L) increases, i.e. it is approaching the maximum value P(s; $\tilde{L}$).

Remark: we know that

$$\left.\frac{\partial \Pi(s)}{\partial s}\right|_{s=\tilde{L}} = \left.\frac{\partial (K-s)}{\partial s}\right|_{s=\tilde{L}} = -1 \tag{18}$$

and

$$\left.\frac{\partial P(s;\tilde{L})}{\partial s}\right|_{s=\tilde{L}} = \beta_1 (K-\tilde{L})\frac{1}{s}\left(\frac{\tilde{L}}{s}\right)^{-\beta_1}\Bigg|_{s=\tilde{L}}$$

$$= \frac{\beta_1 K}{1-\beta_1}\frac{1-\beta_1}{-\beta_1 K} = -1. \tag{19}$$

Thus, it follows that

$$\left.\frac{\partial P(s;\tilde{L})}{\partial s}\right|_{s=\tilde{L}} = \left.\frac{\partial \Pi(s)}{\partial s}\right|_{s=\tilde{L}}. \tag{20}$$

This is known as the *smooth pasting condition* or the *high contact condition*.

Pricing the perpetual maximum option in one stock case

A maximum option is an option whose payoff is the maximum of two or more stocks or assets, e.g.

$$\Pi(z_1, z_2, z_3, z_4) = \max(z_1, z_2, z_3, z_4), \quad z_1, z_2, z_3, z_4 \geq 0.$$

Consider a perpetual American option with payoff function

$$\Pi(z) = \max(K, z), \qquad z \geq 0, \tag{21}$$

where $K > 0$ is the guaranteed price. This payoff function can be regarded as a one stock case of the maximum option which is the maximum of a stock and a positive constant K. Consider the option-exercise strategies of the form:

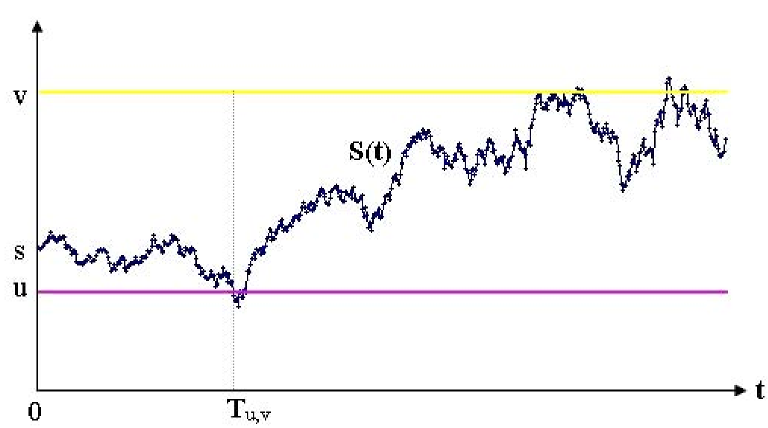
$$T_{u,v} = \min\{t \mid S(t) = u \text{ or } S(t) = v\}, \tag{22}$$

with $0 < u \leq s = S(0) \leq v$. See Figure 5. The strategy $T_{u,v}$ is to exercise the option as soon as the stock price rises to the level v or falls to the level u for the first time; the value of this strategy is

$$V(s; u, v) = E^*[e^{-rT_{u,v}} \Pi(S(T_{u,v})) \mid S(0) = s], \qquad 0 < u \leq s \leq v, \tag{23}$$

where r is the risk-free force of interest.

Figure 5. The Stopping Time $T_{u,v}$



Following Section 10.10 in Panjer, et al. (1998), we express formula (23) as

$$V(s;\, u,\, v) \;=\; \Pi(u)\, A(s;\, u,\, v) \;+\; \Pi(v)\, B(s;\, u,\, v), \qquad 0 < u \le s \le v, \tag{24}$$

where

$$A(s;\, u,\, v) \;=\; E^{*}[e^{-rT_{u,v}}\, I(S(T_{u,v}) = u) \mid S(0) = s], \tag{25}$$

and

$$B(s;\, u,\, v) \;=\; E^{*}[e^{-rT_{u,v}}\, I(S(T_{u,v}) = v) \mid S(0) = s]. \tag{26}$$

Here θ_1 and θ_2 are the solutions of the quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 \;+\; (r \;-\; \frac{\sigma^2}{2} \;-\; \zeta)\theta \;-\; r \;=\; 0, \tag{27}$$

with σ being the diffusion coefficient of the Brownian motion $\{\ln S(t)\}$ and ζ the constant dividend-yield rate. For quadratic equation (27), there are two roots, say θ_1 and θ_2. For $\theta = \theta_1$ and $\theta = \theta_2$, the stochastic process $\{e^{-rt + \theta X(t)}\}_{0 \le t \le T_{u,v}}$ is a bounded martingale. By the optional sampling theorem, we have

$$1 = E^{*}[e^{-rT_{u,v} + \theta X(T_{u,v})}] = E^{*}\left[\left(I(S(T_{u,v}) = u) + I(S(T_{u,v}) = v)\right) e^{-rT_{u,v}} \left(e^{X(T_{u,v})}\right)^{\theta}\right], \tag{28}$$

or

$$1 = E^{*}\left[\, I(S(T_{u,v}) = u)\; e^{-rT_{u,v}} \left(\frac{u}{s}\right)^{\theta}\right] + E^{*}\left[\, I(S(T_{u,v}) = v)\; e^{-rT_{u,v}} \left(\frac{v}{s}\right)^{\theta}\right]. \tag{29}$$

It follows from (25), (26) and (29) that

$$A(s; u, v)\left(\frac{u}{s}\right)^{\theta} + B(s; u, v)\left(\frac{v}{s}\right)^{\theta} = 1. \tag{30}$$

Thus, for roots θ_1 and θ_2, we have

$$A(s; u, v)\left(\frac{u}{s}\right)^{\theta_1} + B(s; u, v)\left(\frac{v}{s}\right)^{\theta_1} = 1 \tag{31}$$

and

$$A(s;u,v)\left(\frac{u}{s}\right)^{\theta_2} + B(s;u,v)\left(\frac{v}{s}\right)^{\theta_2} = 1. \tag{32}$$

From (31) and (32), we obtain

$$A(s;u,v) = \frac{v^{\theta_2}s^{\theta_1} - v^{\theta_1}s^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}} \tag{33}$$

and

$$B(s;u,v) = \frac{s^{\theta_2}u^{\theta_1} - s^{\theta_1}u^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}}. \tag{34}$$

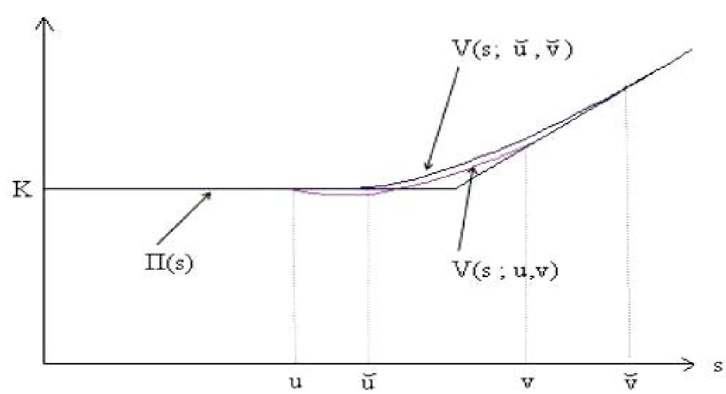
Now, we can substitute the expressions (33) and (34) into the right hand side of (24) to get

$$V(s;u,v) = \Pi(u)\frac{v^{\theta_2}s^{\theta_1} - v^{\theta_1}s^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}} + \Pi(v)\frac{s^{\theta_2}u^{\theta_1} - s^{\theta_1}u^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}}. \tag{35}$$

As shown in Figure 6, V(s; u, v) is the value of the Strategy $T_{u,v}$, which is represented by the red curve between u and v.

Figure 6. The Value of the Strategy $T_{u,v}$ and

the Price of the Perpetual Option



The problem is to find $\breve{u}$ and $\breve{v}$, the value of u and v that maximize V(s; u, v). Then V(s; $\breve{u}$, $\breve{v}$), $\breve{u} \le s = S(0) \le \breve{v}$, is the price of the perpetual American option. The optimal values $\breve{u}$ and $\breve{v}$ are obtained from the first-order conditions:

$$V_u(s;\ \breve{u},\ \breve{v}) = 0, \tag{36}$$

$$V_v(s;\ \breve{u},\ \breve{v}) = 0. \tag{37}$$

We can show that (36) and (37) are equivalent to the *high contact* or *smooth pasting* conditions:

$$V_s(\breve{u};\ \breve{u},\ \breve{v}) = \Pi'(\breve{u}), \tag{38}$$

$$V_s(\breve{v};\ \breve{u},\ \breve{v}) = \Pi'(\breve{v}). \tag{39}$$

Differentiate (24) with respective to u and set s = u, we have

$$V_u(u;\ u,\ v) = \Pi'(u) + \Pi(u)\, A_u(u;\ u,\ v) + \Pi(v)\, B_u(u;\ u,\ v). \tag{40}$$

On the other hand, by differentiating (24) with respective to s and setting s = u, we obtain

$$V_s(u;\ u,\ v) = \Pi(u)\, A_s(u;\ u,\ v) + \Pi(v)\, B_s(u;\ u,\ v). \tag{41}$$

Now let us introduce a new parameter x, where $u < x < s < v$. From (33), by changing some parameters, we can obtain

$$A(s;x,v) = \frac{v^{\theta_2}s^{\theta_1} - v^{\theta_1}s^{\theta_2}}{v^{\theta_2}x^{\theta_1} - v^{\theta_1}x^{\theta_2}}$$

and

$$A(x;u,v) = \frac{v^{\theta_2}x^{\theta_1} - v^{\theta_1}x^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}}.$$

We can see that (33) can be factorized as follow:

$$A(s;\ u,\ v) = A(s;\ x,\ v)\, A(x;\ u,\ v). \tag{42}$$

Similarly, we can also obtain the factorized form of (34) as

$$B(s;\ u,\ v) = A(s;\ x,\ v)\, B(x;\ u,\ v) + B(s;\ x,\ v). \tag{43}$$

By differentiating the identities (42) and (43) with respect to x and setting x = s = u, we have

$$A_u(u;\ u,\ v) + A_s(u;\ u,\ v) = 0, \tag{44}$$

and

$$B_u(u; u, v) + B_s(u; u, v) = 0. \tag{45}$$

A new identity is obtained by combining the identities (40) and (41). By substituting (44) and (45) into the new identity, it can be simplified as

$$V_u(u; u, v) + V_s(u; u, v) = \Pi'(u). \tag{46}$$

Likewise, we can obtain

$$V_v(v; u, v) + V_s(v; u, v) = \Pi'(v). \tag{47}$$

This shows that (36) and (37) are equivalent to (38) and (39).

Now let us solve for $\breve{u}$ and $\breve{v}$. With $\breve{u} < K < \breve{v}$, it follows from (21), (38) and (39) that

$$\frac{\breve{u}}{\breve{v}} = \left(\frac{-\theta_1 (\theta_2 - 1)}{\theta_2 (1 - \theta_1)} \right)^{1/(\theta_2 - \theta_1)}, \tag{48}$$

which is denoted as $\breve{\varphi}$ in Gerber and Shiu (2003). Further, we can determine $\breve{u}$ and $\breve{v}$ in terms of θ_1 and θ_2. We obtain

$$\frac{\breve{v}}{K} = \left(\frac{-\theta_1}{1 - \theta_1} \right)^{-\theta_1/(\theta_2 - \theta_1)} \left(\frac{\theta_2}{\theta_2 - 1} \right)^{\theta_2/(\theta_2 - \theta_1)}, \tag{49}$$

which is denoted as $\breve{c}$ in Gerber and Shiu (2003), and

$$\frac{\breve{u}}{K} = \frac{\breve{v}}{K} \frac{\breve{u}}{\breve{v}} = \left(\frac{-\theta_1}{1 - \theta_1} \right)^{(1-\theta_1)/(\theta_2 - \theta_1)} \left(\frac{\theta_2}{\theta_2 - 1} \right)^{(\theta_2 - 1)/(\theta_2 - \theta_1)}, \tag{50}$$

which is denoted as $\breve{b}$ in Gerber and Shiu (2003). Thus, with $\breve{u} < K < \breve{v}$, for the perpetual American option whose payoff is given by (21), its price is

$$V(s; \breve{u}, \breve{v}) = \begin{cases} K & \text{if } s \le \breve{u} \\ \Pi(\breve{u}) A(s; \breve{u}, \breve{v}) + \Pi(\breve{v}) B(s; \breve{u}, \breve{v}) & \text{if } 0 < \breve{u} < s < \breve{v}, \\ s & \text{if } s \ge \breve{v} \end{cases} \tag{51}$$

$$= \begin{cases} K & \text{if } s \le \breve{u} \\ K \dfrac{\theta_2 \left(s/\breve{u}\right)^{\theta_1} - \theta_1 \left(s/\breve{u}\right)^{\theta_2}}{\theta_2 - \theta_1} & \text{if } 0 < \breve{u} < s < \breve{v}, \\ s & \text{if } s \ge \breve{v} \end{cases} \qquad (52)$$

which is the blue curve between $\breve{u}$ and $\breve{v}$ in Figure 6. For detail derivation of $\breve{u}$ and $\breve{v}$, please refer to Yu (2003).

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