On Optimal Dividends: From Reflection to Refraction

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1. Introduction

What is the optimal dividend strategy, that is, the strategy that maximizes the expectation of the discounted dividends until the possible ruin of a company? De Finetti (1957) formulated the problem and solved it under the assumption that the surplus of the company is a discrete process, with steps of size plus or minus one only. In this model as well as in its continuous counterpart (where the surplus of the company is modeled by a Wiener process), the optimal strategy is a *barrier strategy*. Such a strategy is defined by a positive parameter b, which is the level of the dividend barrier. The modified surplus process is obtained from the original surplus process by *reflection* at the level b, and the dividend stream is the *overflow*. For each given b > 0, the value of the barrier strategy can be calculated explicitly; hence the optimal value of the parameter b can be determined.

Barrier strategies are the solution to a mathematical problem, but the resulting dividend stream is far from practical acceptance. Furthermore, if a barrier strategy is applied, ultimate ruin of the company is certain. These considerations lead to the idea of imposing restrictions on the nature of the dividend stream, resulting in optimization problems with additional constraints.

Jeanblanc-Picqué and Shiryaev (1995) and Asmussen and Taksar (1997) postulated a bounded dividend rate, that is, that the dividends paid per unit time should not exceed an upper bound, which is denoted by α in the following. They show that the optimal dividend strategy is now a *generalized barrier strategy*, which we call a *threshold strategy*. According to such a strategy, dividends are paid at a constant rate α whenever the modified surplus is above the threshold b, and no dividends are paid whenever the modified surplus is below b. Thus the surplus process undergoes what might be called a *stochastic refraction*. Note that a threshold strategy is a *bang-bang strategy*.

The purpose of this note is to present some elementary and down-to-earth calculations in this context. In Sections 2 and 3, closed form expressions for the value of a threshold strategy with an arbitrary parameter b are obtained. Based on these, the optimal value of b is easily obtained in Section 4. Several characterizations of the optimal breakpoint are given in Section 5. In Section 6, the Laplace transform of the time to ruin is derived. If α is less than the drift of the Wiener process, ruin is not certain, and its probability is determined. In the opposite case, the distribution of the total (undiscounted) dividends until ruin is discussed in Section 7. In Section 8, it is shown how the higher order moments and the moment-generating function of the random variable of discounted dividends can be determined.

A review of the literature can be found in Taksar (2001) and Gerber and Shiu (2004a). A recent paper by Boguslavskaya (2003) has generalized the model to the case where the company has a constant salvage value at ruin. Gerber and Shiu (2004b) study the problem in the classical setting – that the aggregate claims are modeled as a compound Poisson process. Li and Garrido (2005) study barrier strategies where the time between successive claims is the sum of a fixed number of independent exponential random variables.

2. The Wiener Process Model and Basic Results

Consider a company with initial surplus or equity $x > 0$. If no dividends were paid, the surplus at time t would be

$$X(t) = x + \mu t + \sigma W(t), \qquad t \geq 0, \tag{2.1}$$

with $\mu > 0$, $\sigma > 0$, and $\{W(t)\}$ being a standard Wiener process. The company will pay dividends to its shareholders. For $t \geq 0$, let $D(t)$ denote the aggregate dividends paid by time t. It is assumed that the payment of dividends has no influence on the business. Thus,

$$\widetilde{X}(t) = X(t) - D(t) \tag{2.2}$$

is the company's surplus at time t. As a reminder that there are dividend payments, we shall call $\widetilde{X}(t)$ the *modified surplus*. Let $\delta > 0$ be the force of interest for valuation, and let D denote the present value of all dividends until ruin,

$$D = \int_0^T e^{-\delta t}\, dD(t), \tag{2.3}$$

where

$$T = \inf\{t \geq 0 \mid \tilde{X}(t) = 0\} \tag{2.4}$$

is the *time of ruin*.

We shall assume that the company pays dividends according to the following strategy governed by parameters $b > 0$ and $\alpha > 0$. Whenever the modified surplus is below the level b, no dividends are paid. However, when the modified surplus is above b, dividends are paid continuously at a constant rate α. Thus the threshold b plays the role of a *break point* or a *regime-switching boundary*. With I(.) denoting the indicator function, an alternative expression for D is

$$D = \alpha \int_0^T e^{-\delta t} I(\tilde{X}(t) > b)\, dt = \alpha\Big[\bar{a}_{\overline{T|}} - \int_0^T e^{-\delta t} I(\tilde{X}(t) < b)\, dt\Big]. \tag{2.5}$$

For $x \geq 0$, we use the symbol V(x; b) to denote the expectation of D,

$$V(x; b) = E[D \mid X(0) = x]. \tag{2.6}$$

For $x \in (0, b)$, V(x; b) satisfies the homogeneous second-order differential equation

$$\frac{\sigma^2}{2}V''(x; b) + \mu V'(x; b) - \delta V(x; b) = 0, \tag{2.7}$$

with the initial condition

$$V(0; b) = 0, \tag{2.8}$$

because $T = 0$ if $x = 0$. It follows that

$$V(x; b) = C(b)(e^{rx} - e^{sx}) \qquad \text{for } 0 \leq x \leq b, \tag{2.9}$$

with the coefficient C(b) being independent of x, and r and s being the roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta = 0. \tag{2.10}$$

We let r denote the positive and s the negative root:

$$r = \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.11}$$

$$s = \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}. \tag{2.12}$$

For $x > b$, the modified surplus process behaves like a Brownian motion with drift $\mu - \alpha$ and variance per unit time σ^2. Now, $V(x; b)$ satisfies the nonhomogeneous second-order differential equation

$$\frac{\sigma^2}{2}V''(x; b) + (\mu - \alpha)V'(x; b) - \delta V(x; b) + \alpha = 0, \tag{2.13}$$

a particular solution of which is α/δ. If there is infinite surplus, then the dividends are a continuous perpetuity of amount α per unit time. Thus we have the condition

$$V(x; b) \to \frac{\alpha}{\delta} \qquad \text{for } x \to \infty. \tag{2.14}$$

It follows that

$$V(x; b) = \frac{\alpha}{\delta} + G(b)e^{ux} \qquad \text{for } x \geq b, \tag{2.15}$$

where the coefficient $G(b)$ is independent of x, and u is the negative root of the characteristic equation of (2.13), namely,

$$u = \frac{-(\mu - \alpha) - \sqrt{(\mu - \alpha)^2 + 2\delta\sigma^2}}{\sigma^2}. \tag{2.16}$$

It is useful to rewrite (2.16) as

$$u = \frac{-2\delta}{(\alpha - \mu) + \sqrt{(\alpha - \mu)^2 + 2\delta\sigma^2}}. \tag{2.17}$$

Using the continuity of the functions V(x; b) and V′(x; b) at x = b, we obtain from (2.9) and (2.15) the conditions:

$$C(b)(e^{rb} - e^{sb}) = \frac{\alpha}{\delta} + G(b)e^{ub}, \tag{2.18}$$

$$C(b)(e^{rb}r - e^{sb}s) = G(b)e^{ub}u, \tag{2.19}$$

from which we can determine the values of the coefficients C(b) and G(b). Multiplying (2.18) by u and subtracting it from (2.19) yields

$$C(b)[e^{rb}(r-u) - e^{sb}(s-u)] = \frac{\alpha}{\delta}(-u).$$

Thus

$$C(b) = \frac{\alpha}{\delta}\frac{-u}{e^{rb}(r-u) + e^{sb}(u-s)}, \tag{2.20}$$

and

$$G(b) = -\frac{\alpha}{\delta}\frac{e^{rb}r - e^{sb}s}{e^{rb}(r-u) + e^{sb}(u-s)}e^{-ub}. \tag{2.21}$$

Hence

$$V(x;b) = \frac{\alpha}{\delta}\frac{(e^{rx} - e^{sx})(-u)}{e^{rb}(r-u) + e^{sb}(u-s)} \qquad \text{for } 0 \le x \le b, \tag{2.22}$$

and

$$V(x;b) = \frac{\alpha}{\delta} - \frac{\alpha}{\delta}\frac{e^{rb}r - e^{sb}s}{e^{rb}(r-u) + e^{sb}(u-s)}e^{u(x-b)} \qquad \text{for } x \ge b. \tag{2.23}$$

Remark The barrier strategy (discussed in Gerber and Shiu 2004a) can be viewed as the limit $\alpha \to \infty$. We see from (2.17) that $u \uparrow 0$ and

$$\lim_{\alpha\to\infty} \alpha u = -\delta. \tag{2.24}$$

It follows from (2.22) and (2.24) that

$$\lim_{\alpha\to\infty} V(x; b) = \frac{e^{rx} - e^{sx}}{e^{rb}r - e^{sb}s} \qquad \text{for } 0 \le x \le b, \tag{2.25}$$

which is (2.11) in Gerber and Shiu (2004a). Now, consider x > b, and rewrite (2.23) as

$$V(x; b) = [V(x; b) - V(b; b)] + V(b; b)$$

$$= \frac{\alpha}{\delta}[1 - e^{u(x-b)}]\frac{e^{rb}r - e^{sb}s}{e^{rb}(r-u) + e^{sb}(u-s)} + \frac{\alpha}{\delta}(-u)\frac{e^{rb} - e^{sb}}{e^{rb}(r-u) + e^{sb}(u-s)}.$$

Then,

$$\lim_{\alpha\to\infty} V(x; b) = (x - b) + \frac{e^{rb} - e^{sb}}{e^{rb}r - e^{sb}s} \qquad \text{for } x > b \tag{2.26}$$

by (2.24). The term (x – b) is the amount of dividends paid instantly at time 0.

3. Alternative Derivation

For X(0) = x ≤ b, the ratio $(e^{rx} - e^{sx})/(e^{rb} - e^{sb})$ is the expected discounted value of a contingent payment of 1, payable as soon as the surplus reaches level b, provided ruin has not yet occurred. See, for example, (2.17) in Gerber and Shiu (2004a). Thus, we have the formula

$$V(x; b) = \frac{e^{rx} - e^{sx}}{e^{rb} - e^{sb}}V(b; b) \qquad \text{for } 0 \le x \le b, \tag{3.1}$$

which is consistent with (2.9).

For X(0) = x > b, let τ be the time when the modified surplus drops to the level b for the first time. Then

$$V(x; b) = E[\alpha\, \bar{a}_{\overline{\tau}|} + V(b; b)e^{-\delta\tau}] = \frac{\alpha}{\delta} - [\frac{\alpha}{\delta} - V(b; b)]E[e^{-\delta\tau}].$$

Because $E[e^{-\delta\tau}] = e^{u(x-b)}$, we have

$$V(x; b) = \frac{\alpha}{\delta} - [\frac{\alpha}{\delta} - V(b; b)]e^{u(x-b)} \qquad \text{for } x \geq b, \tag{3.2}$$

which is consistent with (2.23).

To derive the value of V(b; b), we use the condition that V(x; b) is continuously differentiable. From $V'(b-; b) = V'(b+; b)$, we have

$$\frac{e^{rb}r - e^{sb}s}{e^{rb} - e^{sb}}V(b; b) = [\frac{\alpha}{\delta} - V(b; b)](-u),$$

or

$$V(b; b) = \frac{\alpha}{\delta}\frac{(e^{rb} - e^{sb})(-u)}{e^{rb}(r-u) + e^{sb}(u-s)}. \tag{3.3}$$

4. Optimal Threshold

For given dividend rate $\alpha > 0$, let b* be the optimal value of b, that is, the value that maximizes V(x; b). That this value does not depend on the initial surplus x can be seen as follows. From (2.9) and (2.15) we see that maximizing V(x; b) means maximizing C(b) and G(b), respectively. That C(b) and G(b) can be maximized simultaneously follows from the following relation between their derivatives,

$$\frac{dC(b)}{db}(e^{rb} - e^{sb}) = \frac{dG(b)}{db}e^{ub}, \tag{4.1}$$

which is obtained by differentiating (2.18) with respect to b and using (2.19) for a cancellation. Setting the derivative of the denominator in (2.22) with respect to b equal to 0, we obtain

$$b^* = \frac{1}{r-s}\ln\left(\frac{s^2 - us}{r^2 - ur}\right) \tag{4.2}$$

as a preliminary result.

It seems that the higher the dividend rate α, the higher the optimal threshold b^* need to be. We now verify this by showing the derivative $db^*/d\alpha$ is positive. The value b^* is a function of α through u, which is defined by (2.16). Let us write

$$u = u(\alpha). \tag{4.3}$$

From (2.16) and (2.12), we see that

$$u(0) = s \tag{4.4}$$

and that $u(\alpha)$ is an increasing function of α. Thus $u' > 0$. Differentiating (4.2), we have by the chain rule

$$\begin{aligned}\frac{db^*}{d\alpha} &= \frac{1}{r-s}\left(\frac{-su'}{s^2-us} - \frac{-ru'}{r^2-ur}\right)\\ &= \frac{1}{r-s}\left(\frac{-1}{s-u} - \frac{-1}{r-u}\right)u'\\ &= \frac{u'}{(r-u)(u-s)},\end{aligned} \tag{4.5}$$

which is indeed positive for $\alpha > 0$.

The expression on the right-hand side of (4.2) can be negative. It is 0 for

$$u = r + s = -2\mu/\sigma^2. \tag{4.6}$$

Applying this condition to (2.16), we find that the right-hand side of (4.2) vanishes if

$$2\mu\alpha = \delta\sigma^2. \tag{4.7}$$

Let us write

$$R = \frac{2\mu}{\sigma^2} \tag{4.8}$$

to emphasize its correspondence with the *adjustment coefficient* in classical risk theory. It follows from (4.7) that the optimal value of b is given by (4.2) if

$$\frac{\alpha}{\delta} > \frac{1}{R}. \tag{4.9}$$

If condition (4.9) is violated, i.e., if

$$\frac{\alpha}{\delta} \leq \frac{1}{R}, \tag{4.10}$$

the optimal value of b is 0. Then the expected present value of dividends is

$$V(x; 0) = \frac{\alpha}{\delta}(1 - e^{ux}) \tag{4.11}$$

by (2.23). This formula follows also from the observation that the dividend stream is constant between time 0 and the time of ruin, and hence it can be evaluated as the difference between a perpetuity and a deferred perpetuity. With $\alpha = 1$, formula (4.11) corresponds to the well-known life contingencies formula

$$\bar{a}_y = \frac{1}{\delta}(1 - \bar{A}_y).$$

5. Discussion of the Optimal Threshold

Throughout this section we assume that α is sufficiently large, so that (4.9) holds and the optimal value of b is given by (4.2).

The optimal threshold b^* can be characterized by the condition that the second derivative $V''(x; b)$ is continuous at $x = b$. Thus

$$V''(b+; b) = V''(b-; b) \tag{5.1}$$

if and only if $b = b^*$. This condition is known as a *high contact condition* in finance literature and a *smooth pasting condition* in literature on optimal stopping. To see this, observe that the k-th derivatives of (2.22) and (2.23) with respect to x are

$$V^{(k)}(x;\, b) \;=\; \frac{\alpha}{\delta}\,\frac{(e^{rx}r^k - e^{sx}s^k)(-u)}{e^{rb}(r-u) \,+\, e^{sb}(u-s)} \qquad \text{for } x < b, \tag{5.2}$$

and

$$V^{(k)}(x;\, b) \;=\; -\frac{\alpha}{\delta}\,\frac{e^{rb}r - e^{sb}s}{e^{rb}(r-u) \,+\, e^{sb}(u-s)}\,e^{u(x-b)}u^k \qquad \text{for } x > b, \tag{5.3}$$

respectively. Thus, (5.1) holds if and only if

$$(e^{rb}r^2 \,-\, e^{sb}s^2) \;=\; (e^{rb}r \,-\, e^{sb}s)u,$$

which holds if and only $b = b^*$ as given in (4.2),

There is a second characterization of the optimal threshold b^*. To obtain it, we set $x = b-$ in the differential equation (2.7) and $x = b+$ in (2.13). Taking their difference yields the formula

$$V'(b;\, b) \;=\; 1 \;+\; \frac{\sigma^2}{2\alpha}[V''(b+;\, b) \;-\; V''(b-;\, b)]. \tag{5.4}$$

From this and the first characterization it follows that

$$V'(b;\, b) \;=\; 1 \tag{5.5}$$

if and only if $b = b^*$.

This second characterization is somewhat surprising because in the case of a barrier strategy, condition (5.5) holds for *all* b; see (2.25) and (2.26). Also, it thus follows from (2.9) that

$$V(x;\, b^*) \;=\; \frac{e^{rx} - e^{sx}}{re^{rb^*} - se^{sb^*}}, \qquad 0 \le x \le b^*. \tag{5.6}$$

By comparing (5.6) with (2.26), we find the following *astonishing* result: Consider the threshold strategy with optimal break point b^*. Then for $0 \le x \le b^*$, the expected value of D is identical to the expected value of the discounted dividends under the barrier strategy ($\alpha = \infty$) with parameter equal b^*.

Remarks (i) In the literature, there are alternative expressions for b*. Applying (5.2), with k = 1, to condition (5.5), with b = b*, yields

$$\frac{\alpha}{\delta}\,\frac{(e^{rb^*}r - e^{sb^*}s)(-u)}{e^{rb^*}(r-u) + e^{sb^*}(u-s)} = 1,$$

or

$$\frac{\alpha}{\delta}[e^{(r-s)b^*}r - s] = \frac{1}{-u}[e^{(r-s)b^*}(r-u) + (u-s)].$$

With the definition

$$q = \frac{\alpha}{\delta} + \frac{1}{u}, \tag{5.7}$$

we obtain

$$b^* = \frac{1}{r-s}\ln\left(\frac{1-qs}{1-qr}\right). \tag{5.8}$$

This alternative expression for b* is (2.26) of Asmussen and Taksar (1997). Formula (2.27) of Jeanblanc-Picqué and Shiryaev (1995) gives an expression for the hyperbolic tangent of b*(r – s)/2.

(ii) From (2.15), we see that

$$V(x; b) = \frac{\alpha}{\delta} + \frac{1}{u}V'(x; b) \qquad \text{for } x > b. \tag{5.9}$$

Setting b = b* and x = b*, and using condition (5.5) with b = b*, we obtain

$$V(b^*; b^*) = \frac{\alpha}{\delta} + \frac{1}{u}\cdot 1 = q. \tag{5.10}$$

Thus q is the maximal value of the discounted dividends until ruin if the initial surplus is b*. Now, it follows from (3.2) and (5.10) that

$$V(x; b^*) \;=\; \frac{\alpha}{\delta} \;+\; \frac{1}{u}e^{u(x-b^*)} \qquad\qquad \text{for } x \geq b^*, \tag{5.11}$$

and from (3.1) and (5.10) that

$$V(x; b^*) \;=\; \frac{e^{rx} - e^{sx}}{e^{rb^*} - e^{sb^*}}q \qquad\qquad \text{for } 0 \leq x \leq b^*. \tag{5.12}$$

Formulas (5.11) and (5.12) are (2.33) in Jeanblanc-Picqué and Shiryaev (1995) and (2.28) in Asmussen and Taksar (1997). It is interesting to rewrite (5.11) as

$$V(x; b^*) \;=\; q \;+\; \bar{a}_{\overline{x-b^*|}} \;=\; V(b^*; b^*) \;+\; \bar{a}_{\overline{x-b^*|}} \qquad \text{for } x \geq b^*, \tag{5.13}$$

where the annuity is evaluated at the force of interest $-u$. Note that (5.11) and the continuity of $V''(x; b^*)$ at $x = b^*$ show that

$$V''(b^*; b^*) \;=\; u. \tag{5.14}$$

(iii) It follows from (2.7) and the two characterizations of b^* that

$$\frac{\sigma^2}{2}V''(b^*; b^*) \;+\; \mu \;-\; \delta V(b^*; b^*) \;=\; 0. \tag{5.15}$$

Applying (5.14) to (5.15) yields

$$V(b^*; b^*) \;=\; \frac{\mu}{\delta} \;+\; \frac{u\sigma^2}{2\delta}, \tag{5.16}$$

which must be another expression for q.

(iv) Consider the limit $\alpha \to \infty$. It follows from (4.2) that

$$\lim_{\alpha\to\infty} b^* \;=\; \frac{1}{r-s}\ln\left(\frac{s^2}{r^2}\right) \;=\; \frac{2}{r-s}\ln\left(\frac{-s}{r}\right), \tag{5.17}$$

which is (10.2) in Gerber (1972) and (5.2) in Gerber and Shiu (2004a). In Section 2, we have noted that $u \uparrow 0$ as $\alpha \to \infty$. Thus, from (5.16) we immediately obtain

$$\lim_{\alpha\to\infty} V(b^*; b^*) \;=\; \frac{\mu}{\delta}, \tag{5.18}$$

which has been obtained by Gerber (1972). With $\alpha = \infty$ and $b^* < \infty$, ruin is certain. However, μ/δ is identical to the present value of a perpetuity with continuous payments at a rate of μ. The intriguing formula (5.18) also follows from (5.10) and the result

$$\lim_{\alpha\to\infty} q = \frac{\mu}{\delta}. \tag{5.19}$$

Finally, we note that (5.19) implies $q/\alpha \to 0$, which is equivalent to (2.24).

6. The Distribution of T under a Threshold Strategy

Consider that the threshold strategy with threshold b being applied. We are interested in the distribution of the time of ruin, T. In this section, we calculate

$$L(x; b) = E[e^{-\delta T} \mid X(0) = x], \tag{6.1}$$

where $x = X(0)$ is the initial surplus or capital. This is the expected present value of a payment of 1 at the time of ruin, and at the same time, the *Laplace transform* of the probability density function of T.

As a function of the initial surplus x, $0 < x < b$, $L(x; b)$ satisfies the homogeneous second-order differential equations

$$\frac{\sigma^2}{2}L''(x; b) + \mu L'(x; b) - \delta L(x; b) = 0 \qquad \text{for } 0 < x < b, \tag{6.2}$$

and

$$\frac{\sigma^2}{2}L''(x; b) + (\mu - \alpha)L'(x; b) - \delta L(x; b) = 0 \qquad \text{for } x > b. \tag{6.3}$$

If $X(0) = x = \infty$, then $T = \infty$. Thus we have the condition

$$\lim_{x\to\infty} L(x; b) = 0. \tag{6.4}$$

Subject to (6.4), the solution of (6.3) is

$$L(x; b) = e^{u(x-b)}L(b; b), \qquad x \geq b, \tag{6.5}$$

where u is given by (2.16), the negative root of the characteristic equation of (6.3). Formula (6.5) can be understood in terms of the time decomposition, $T = \tau + (T - \tau)$, where the stopping time τ was defined in Section 3.

If $X(0) = x = 0$, then $T = 0$. Thus

$$L(0; b) = 1. \tag{6.6}$$

Subject to condition (6.6), the solution of (6.2) is

$$L(x; b) = e^{sx} + a(e^{rx} - e^{sx}), \tag{6.7}$$

where r and s are given by (2.11) and (2.12), respectively, and the coefficient a is determined by the continuity of the functions $L(x; b)$ and $L'(x; b)$ at $x = b$:

$$e^{sb} + a(e^{rb} - e^{sb}) = L(b; b), \tag{6.8}$$

$$e^{sb}s + a(e^{rb}r - e^{sb}s) = L(b; b)u. \tag{6.9}$$

Multiplying (6.8) with u and subtracting it from (6.9) yields

$$e^{sb}(s-u) + a[e^{rb}(r-u) - e^{sb}(s-u)] = 0. \tag{6.10}$$

Thus,

$$a = \frac{(u-s)e^{sb}}{(u-s)e^{sb} + (r-u)e^{rb}}. \tag{6.11}$$

From this and (6.7), we obtain the Laplace transform of T for $0 \leq x \leq b$:

$$L(x; b) = \frac{(u-s)e^{sb+rx} + (r-u)e^{rb+sx}}{(u-s)e^{sb} + (r-u)e^{rb}} = \frac{(u-s)e^{-r(b-x)} + (r-u)e^{-s(b-x)}}{(u-s)e^{-rb} + (r-u)e^{-sb}}. \tag{6.12}$$

In particular,

$$L(b; b) = \frac{r-s}{(r-u)e^{-sb} + (u-s)e^{-rb}}, \tag{6.13}$$

which is needed for evaluating (6.5).

Remarks (i) In the limit $\alpha \to \infty$, we have $u = 0$. Then (6.12) is (3.7) in Gerber and Shiu (2004a) and can be found in Cox and Miller (1965, p. 233, Example 5.6).

(ii) Since $\delta > 0$,

$$E[e^{-\delta T} \mid X(0) = x] = E[e^{-\delta T} I(T < \infty) \mid X(0) = x]. \tag{6.14}$$

Thus,

$$\lim_{\delta \downarrow 0} L(x; b) = E[I(T < \infty) \mid X(0) = x] = \Pr(T < \infty \mid X(0) = x) = \psi(x), \tag{6.15}$$

the probability of ruin. If $\alpha \geq \mu$, ruin is certain. Hence we now assume $\alpha < \mu$. It follows from (2.11), (2.12) and (2.16) that

$$\lim_{\delta \to 0} r = 0, \tag{6.16}$$

$$\lim_{\delta \to 0} s = -\frac{2\mu}{\sigma^2} = -R, \tag{6.17}$$

and

$$\lim_{\delta \to 0} u = -\frac{2(\mu - \alpha)}{\sigma^2} = -R + \frac{2\alpha}{\sigma^2}, \tag{6.18}$$

respectively. Thus (6.12) and (6.5) become

$$\psi(x) = \frac{\alpha + (\mu - \alpha)e^{R(b-x)}}{\alpha + (\mu - \alpha)e^{Rb}} \quad \text{for } 0 \leq x \leq b, \tag{6.19}$$

and

$$\psi(x) = e^{-2(\mu-\alpha)(x-b)/\sigma^2}\psi(b) = \frac{\mu e^{-2(\mu-\alpha)(x-b)/\sigma^2}}{\alpha + (\mu - \alpha)e^{Rb}} \quad \text{for } x > b, \tag{6.20}$$

respectively.

(iii) If $\alpha = 0$ or if $b = \infty$, then (6.19) simplifies as

$$\psi(x) = e^{-Rx}. \tag{6.21}$$

This is a well-known result; see, for example, Corollary 8.25 in Klugman, Panjer and Willmot (2004).

(iv) Following Deprez (2004), we note that, for $X(0) = x$ and $0 < w < x \le b$,

$$V(x; b) = V(x - w; b - w) + L(x - w; b - w)V(w; b). \tag{6.22}$$

Thus,

$$L(x - w; b - w) = \frac{V(x;b) - V(x - w;b - w)}{V(w;b)}, \tag{6.23}$$

which, with (2.22), yields another way to calculate the Laplace transform L.

(v) By (2.5), another relation between the functions V and L is

$$V(x; b) = \frac{\alpha}{\delta}[1 - L(x; b)] - \alpha E[\int_0^T v^t I(\tilde{X}(t) < b)\, dt]. \tag{6.24}$$

(vi) The situation where dividend payments do not end with ruin is of some mathematical interest. Let $W(x; b)$, $-\infty < x < \infty$, denote the expectation of the present value of all dividends. Then, by considering

$$D = \alpha \int_0^\infty e^{-\delta t} I(\tilde{X}(t) > b)\, dt - \alpha \int_T^\infty e^{-\delta t} I(\tilde{X}(t) > b)\, dt,$$

we have

$$V(x; b) = W(x; b) - L(x; b)W(0; b), \qquad x \ge 0. \tag{6.25}$$

The function $W(x; b)$ satisfies the differential equation (2.7), but for $-\infty < x < b$. Because $W(-\infty; b) = 0$, it follows that

$$W(x; b) = \kappa(b)e^{rx}, \qquad -\infty < x \le b.$$

Similarly,

$$W(x; b) = \frac{\alpha}{\delta} + \gamma(b)e^{ux}, \qquad x \ge b.$$

The coefficients $\kappa(b)$ and $\gamma(b)$ are independent of x and are determined from the smooth junction conditions:

$$W(b-; b) = W(b+; b),$$

$$W'(b-; b) = W'(b+; b).$$

This way, one finds that

$$W(x; b) = \frac{-u}{r-u}\frac{\alpha}{\delta}e^{-r(b-x)}, \qquad -\infty < x \leq b, \qquad (6.26)$$

and

$$W(x; b) = \frac{\alpha}{\delta} - \frac{r}{r-u}\frac{\alpha}{\delta}e^{u(b-x)}. \qquad x \geq b. \qquad (6.27)$$

The reader may now find it instructive to verify (2.22) and (2.23) by means of (6.25).

7. The Distribution of D(T)

If $0 < \alpha < \mu$, ruin does not occur with positive probability $1 - \psi(x)$, and therefore the aggregate dividends are infinite with positive probability. Hence we assume $\alpha \geq \mu$, so that D(T) is finite with certainty. Our first goal is to determine

$$M(x, y; b) = E[e^{yD(T)} \mid X(0) = x], \qquad (7.1)$$

the moment-generating function of D(T). To avoid the question of its existence, we consider (7.1) for $y < 0$.

As a function of x, the moment-generating function M(x, y; b) satisfies the homogeneous *ordinary* differential equations

$$\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}M(x, y; b) + \mu\frac{\partial}{\partial x}M(x, y; b) = 0 \qquad \text{for } 0 < x < b, \qquad (7.2)$$

and

$$\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}M(x, y; b) + (\mu - \alpha)\frac{\partial}{\partial x}M(x, y; b) + \alpha y M(x, y; b) = 0 \quad \text{for } x > b. \quad (7.3)$$

If $X(0) = x = \infty$, then $T = \infty$ and $D(T) = \infty$. Thus we have the condition

$$\lim_{x\to\infty} M(x, y; b) = 0, \quad (7.4)$$

subject to which, the solution of (7.3) is

$$M(x, y; b) = M(b, y; b)\, e^{v(x-b)}, \quad (7.5)$$

where

$$v = \frac{-(\mu-\alpha) - \sqrt{(\mu-\alpha)^2 - 2\alpha y\sigma^2}}{\sigma^2} \quad (7.6)$$

is the negative root of the characteristic equation of (7.3).

If $X(0) = x = 0$, then $T = 0$ and $D(T) = 0$. Thus

$$M(0, y; b) = 1. \quad (7.7)$$

Subject to condition (7.7), the solution of (7.2) is

$$M(x, y; b) = 1 - a(1 - e^{-Rx}), \quad (7.8)$$

where the coefficient a is determined by the continuity of the functions $M(x, y; b)$ and

$\frac{\partial}{\partial x}M(x, y; b)$ at $x = b$:

$$1 - a(1 - e^{-Rb}) = M(b, y; b), \quad (7.9)$$

$$-a R\, e^{-Rb} = M(b, y; b)\, v. \quad (7.10)$$

These two equations yield

$$a = \frac{1}{1-(1+\frac{R}{v})e^{-Rb}}, \quad (7.11)$$

applying which to (7.8), we obtain

$$M(x, y; b) = 1 - \frac{1-e^{-Rx}}{1-(1+\frac{R}{v})e^{-Rb}}$$

$$= \frac{(v+R)e^{-Rb} - ve^{-Rx}}{(v+R)e^{-Rb} - v} \quad \text{for } 0 \le x \le b. \tag{7.12}$$

Define

$$\bar{s}_{\overline{x|}} = \frac{e^{Rx}-1}{R}; \tag{7.13}$$

in this "actuarial" definition, R takes the role of a force of interest. Then, formula (7.12) can be written as

$$M(x, y; b) = \frac{1-v\bar{s}_{\overline{b-x|}}}{1-v\bar{s}_{\overline{b|}}} \qquad \text{for } 0 \le x \le b. \tag{7.14}$$

In particular,

$$M(b, y; b) = \frac{1}{1-v\bar{s}_{\overline{b|}}}, \tag{7.15}$$

applying which to (7.5) yields

$$M(x, y; b) = \frac{e^{v(x-b)}}{1-v\bar{s}_{\overline{b|}}} \qquad \text{for } x > b. \tag{7.16}$$

As a check for formula (7.14), we consider $\alpha \to \infty$. We see from formula (7.6) that $v \to y$. Hence, for $0 \le x \le b$,

$$\lim_{\alpha\to\infty} M(x, y; b) = \frac{1-y\bar{s}_{\overline{b-x|}}}{1-y\bar{s}_{\overline{b|}}}, \tag{7.17}$$

which is (6.2) in Gerber and Shiu (2004a).

Consider now the case $0 < x < b$. Then D(T) is a compound geometric random variable:

$$D(T) = D_1 + D_2 + \dots + D_N. \tag{7.18}$$

Here N is the number of times until ruin that the modified surplus returns to the initial level x after a visit at the threshold b, and D_n represents the total dividends paid between the (n–1) th and n th return to the level x. It is well known (see, for example, Gerber and Shiu 2004a, formula 5.3) that for X(0) = x, the probability that ruin occurs before the threshold b is attained is

$$p = \frac{e^{R(b-x)}-1}{e^{Rb}-1} = \frac{\bar{s}_{\overline{b-x}|}}{\bar{s}_{\overline{b}|}}. \tag{7.19}$$

That D(T) has a compound geometric distribution can be confirmed directly: Compare (7.14) with (A.5) in the Appendix and note that M(x, y; b) depends on y through v given in (7.6). Moreover, it follows from (A.2) that the common moment-generating function of D_n's is

$$\frac{1}{1-v\bar{s}_{\overline{b-x}|}} = \frac{\sigma^2}{\sigma^2+[(\mu-\alpha)+\sqrt{(\mu-\alpha)^2-2\alpha y\sigma^2}\,]\bar{s}_{\overline{b-x}|}}. \tag{7.20}$$

In the limit $\alpha \to \infty$, v = y, and (7.20) is the moment-generating function of an exponential random variable with mean $\bar{s}_{\overline{b-x}|}$.

In the special case of $\alpha = \mu$, formula (7.6) simplifies as

$$v = \frac{-\sqrt{-2\mu y\sigma^2}}{\sigma^2} = -\sqrt{-yR}\,. \tag{7.21}$$

Thus (7.14) becomes

$$M(x, y; b) = \frac{1+\sqrt{-yR}\,\bar{s}_{\overline{b-x}|}}{1+\sqrt{-yR}\,\bar{s}_{\overline{b}|}} \qquad \text{for } 0 \le x \le b, \tag{7.22}$$

while (7.20) reduces to

$$\frac{1}{1+\sqrt{-yR}\,\bar{s}_{\overline{b-x}|}}. \tag{7.23}$$

Remark Because $y < 0$, the formulas for $M(x, y; b)$ are also valid if $0 < \alpha < \mu$, where $D(T) = \infty$ with positive probability. Then $e^{yD(T)}$ in (7.1) has the value 0 if $T = \infty$. It follows that

$$\lim_{y \uparrow 0} M(x, y; b) = \Pr(T < \infty) = \psi(x). \tag{7.24}$$

From this, the relation

$$\lim_{y \uparrow 0} v = -\frac{2(\mu - \alpha)}{\sigma^2}, \tag{7.25}$$

and formulas (7.12) and (7.16), we can retrieve formulas (6.19) and (6.20), respectively.

8. The Moments and the Moment-Generating Function of D

Let

$$M(x, y; b) = E[e^{yD} \mid X(0) = x] \tag{8.1}$$

denote the moment-generating function of D. In Section 7, the case $\delta = 0$ is discussed. We assume $\delta > 0$. Then $0 \le D \le \alpha/\delta$, and $M(x, y; b)$ exists for all y.

For $0 < x < b$, the moment-generating function M satisfies the partial differential equation

$$\frac{\sigma^2}{2}\frac{\partial^2 M}{\partial x^2} + \mu\frac{\partial M}{\partial x} - \delta y\frac{\partial M}{\partial y} = 0, \tag{8.2}$$

which is the same as (4.3) in Gerber and Shiu (2004a) and generalizes (7.2) above. For $x > b$, we have

$$\frac{\sigma^2}{2}\frac{\partial^2 M}{\partial x^2} + (\mu - \alpha)\frac{\partial M}{\partial x} + \alpha y M - \delta y\frac{\partial M}{\partial y} = 0, \tag{8.3}$$

which generalizes (7.3). The boundary conditions are (7.7) and

$$\lim_{x\to\infty} M(x, y; b) = e^{y\alpha/\delta} \tag{8.4}$$

because $\lim_{x\to\infty} D = \alpha/\delta$, the present value of a continuous perpetuity of rate α. Finally, as functions of x, M(x, y; b) and $\frac{\partial}{\partial x} M(x, y; b)$ are continuous at the junction $x = b$.

We set

$$M(x, y; b) = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!} V_k(x; b), \tag{8.5}$$

where

$$V_k(x; b) = E[D^k \mid X(0) = x] \tag{8.6}$$

is the k-th moment of D. Substitution of (8.5) in (8.2) and (8.3), with subsequent comparison of the coefficients of y^k, yields the ordinary differential equations

$$\frac{\sigma^2}{2} V_k''(x; b) + \mu V_k'(x; b) - \delta k V_k(x; b) = 0, \tag{8.7}$$

for $0 < x < b$, and

$$\frac{\sigma^2}{2} V_k''(x; b) + (\mu - \alpha) V_k'(x; b) - \delta k V_k(x; b) + \alpha k V_{k-1}(x; b) = 0, \tag{8.8}$$

for $x > b$. They generalize (2.7) and (2.13), which are for $k = 1$. The boundary conditions are

$$V_k(0; b) = 0 \tag{8.9}$$

and

$$\lim_{x\to\infty} V_k(x; b) = \left(\frac{\alpha}{\delta}\right)^k. \tag{8.10}$$

We shall show how $V_k(x; b)$ can be determined recursively with respect to k.

From (8.7) and (8.9), it follows that

$$V_k(x; b) = C_k(b)\left(e^{r_k x} - e^{s_k x}\right), \tag{8.11}$$

where $r_k > 0$ and $s_k < 0$ are the solutions of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta k = 0, \tag{8.12}$$

and $C_k(b)$, which does not depend on x, has yet to be determined. The solution of (8.8) and (8.10) is of the form

$$V_k(x; b) = \left(\frac{\alpha}{\delta}\right)^k + \sum_{j=1}^{k} G_{j,k}(b)\, e^{u_j(x-b)}, \tag{8.13}$$

$x \geq b$, with u_j being the negative solution of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + (\mu - \alpha)\xi - \delta j = 0. \tag{8.14}$$

Note that $r_1 = r$, $s_1 = s$, $u_1 = u$, and $C_1(b) = C(b)$ and $G_{1,1}(b) = G(b)$ are given in (2.20) and (2.21), respectively. Substituting (8.13) and

$$V_{k-1}(x; b) = \left(\frac{\alpha}{\delta}\right)^{k-1} + \sum_{j=1}^{k-1} G_{j,k-1}(b)\, e^{u_j(x-b)} \tag{8.15}$$

in (8.8) and comparing the coefficients of $e^{u_j(x-b)}$ yields the equation

$$\left[\frac{\sigma^2}{2}\xi^2 + (\mu - \alpha)\xi - \delta k\right] G_{j,k}(b) + \alpha k\, G_{j,k-1}(b) = 0. \tag{8.16}$$

From this and the fact that u_j is a solution of (8.14), we obtain the recursion

$$G_{j,k}(b) = \frac{\alpha k}{\delta(k-j)} G_{j,k-1}(b) \tag{8.17}$$

for $j = 1, 2, \ldots, k-1$. Finally, $C_k(b)$ and $G_{k,k}(b)$ are determined from the condition that $V_k(x; b)$ and $V_k'(x; b)$ are continuous at $x = b$.

From (8.17) it follows that

$$G_{j,k}(b) = \left(\frac{\alpha}{\delta}\right)^{k-j} \binom{k}{j} G_{j,j}(b) \tag{8.18}$$

for k = j, j+1, j+2, … From this and (8.13), we obtain the formula

$$V_k(x; b) = \left(\frac{\alpha}{\delta}\right)^{k} + \sum_{j=1}^{k} \left(\frac{\alpha}{\delta}\right)^{k-j} \binom{k}{j} G_{j,j}(b)\, e^{u_j(x-b)} \tag{8.19}$$

for $x \geq b$.

If we substitute (8.19) in (8.5), we obtain after simplification the formula

$$M(x, y; b) = e^{y\alpha/\delta} + e^{y\alpha/\delta} \sum_{j=1}^{\infty} \frac{y^j}{j!} G_{j,j}(b)\ e^{u_j(x-b)} \tag{8.20}$$

for $x \geq b$. It is instructive to verify directly that this function satisfies the partial differential equation (8.3).

Acknowledgment

We gratefully acknowledge the support from the Patrick Poon Lecture Series in Actuarial Science, Principal Financial Group Foundation, and Robert J. Myers, FCA, FCAS, FSA.

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Appendix

This Appendix presents some equivalent expressions for the moment-generating function of a compound geometric random variable,

$$S = 0 \qquad\qquad \text{if } N = 0,$$

$$S = X_1 + X_2 + \ldots + X_N \qquad\qquad \text{if } N \geq 1. \tag{A.1}$$

Let p and q (p + q = 1) be the parameters of the geometric distribution, Pr(N = 0) = p.

Let the moment-generating function of each summand, X, be

$$M_X(y) = \frac{1}{1 - g(y)}. \tag{A.2}$$

Then the moment-generating function of S is

$$\begin{aligned} M_S(y) &= p\sum_{k=0}^{\infty}\left(\frac{q}{1-g(y)}\right)^k \\ &= \frac{p}{1 - \dfrac{q}{1-g(y)}}. \end{aligned} \tag{A.3}$$

Thus

$$M_S(y) = \frac{p[1 - g(y)]}{p - g(y)} = \frac{1 - g(y)}{1 - \dfrac{g(y)}{p}}. \tag{A.4}$$

Hence, if a distribution has a moment-generating function of the form

$$M(y) = \frac{1 - g(y)}{1 - \beta g(y)}, \tag{A.5}$$

with $\beta > 1$, we can conclude by comparing (A.5) with (A.4) and (A.3) that it is a compound geometric distribution. The geometric distribution has parameter $p = 1/\beta$, and the moment-generating function of each summand is given by (A.2).

Finally, writing (A.4) as

$$M_S(y) = p + q \frac{1}{1 - \frac{g(y)}{p}}, \tag{A.6}$$

we see that the underlying distribution of S is a mixture of the degenerate distribution at 0 and a distribution with moment-generating function

$$\frac{1}{1 - \frac{g(y)}{p}}. \tag{A.7}$$