SOCIETY OF ACTUARIES

EXAM M ACTUARIAL MODELS

ADDITIONAL EXAM M SAMPLE QUESTIONS AND SOLUTIONS

1. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Given that an insured is classified as Preferred at the start of the second year, find the probability that that insured is Preferred at the start of the fourth year.

Question 1 solution

This is the probability ${}_2Q_1^{(1,1)}$, which is just the $(1,1)$-entry of

$$Q_1 Q_2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.73333 & 0.26667 \\ 0.33333 & 0.66667 \end{bmatrix},$$

namely $0.75(0.73333) + 0.25(0.33333) = 0.63333$.

2. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Given that an insured is classified as Preferred at the start of the second year, find the probability that that insured transitions from being Preferred at the start of the fourth year to being Standard at the start of the fifth year.

Question 2 solution

This can be computed as ${}_2Q_1^{(1,1)}Q_3^{(1,2)}$. ${}_2Q_1^{(1,1)}$ is just the $(1,1)$-entry of

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.73333 & 0.26667 \\ 0.33333 & 0.66667 \end{bmatrix},$$

namely $0.75(0.73333) + 0.25(0.33333) = 0.63333$. So the answer is $(0.63333)(0.275) = 0.17417$.

3. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Given that an insured is classified as Standard at the start of the second year, find the probability that that insured remains Standard at the start of each of the next three years.

Question 3 solution

This probability is just ${}_3P_2^{(2)} = Q_1^{(2,2)} Q_2^{(2,2)} Q_3^{(2,2)} = (0.7)(0.66667)(0.65) = 0.30333.$

4. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Driver A is Standard now, at the start of the fourth year. For $k = 0, 1$, there is a cost of $10(1.1)^k$ at the end of year $4+k$ for a transition from Standard at the start of that year to Preferred at the start of the next year. Find the actuarial present value now of these cash flows for Driver A using 15% interest.

Question 4 solution

The triple-product summation for this actuarial present value is $Q_3^{(2,1)}(10)(\frac{1}{1.15}) + Q_3^{(2,2)}Q_4^{(2,1)}[10(1.1)](\frac{1}{1.15^2}) = \frac{(0.35)(10)}{1.15} + \frac{(0.65)(0.36)[10(1.1)]}{1.15^2} = 4.9898$.

5. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Driver B is Preferred now, at the start of the fourth year. For $k = 0, 1$, there is a cost of $10(1.1)^k$ at the end of year $4+k$ for a transition from Standard at the start of that year to Preferred at the start of the next year. Find the actuarial present value now of these cash flows for Driver B using 15% interest.

Question 5 solution

The triple-product summation for this actuarial present value is

$$[Q_3^{(1,2)}Q_4^{(2,1)}][10(1.1)](\frac{1}{1.15^2}) = \frac{(0.275)(0.36)(11)}{1.15^2} = 0.82344$$

6. A non-homogeneous Markov Chain has transition-probability matrices $\boldsymbol{Q}_n$ and cash-flow matrices ${}_{\ell+1}\boldsymbol{C}$ defining cash flows at time $\ell+1$ for transitions from states at time ℓ to states at time $\ell+1$. You are given that

$$\boldsymbol{Q}_3 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}, \qquad {}_4\boldsymbol{C} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \qquad i = 25\%.$$

You also are given that the actuarial present value at time 4 of future cash flows at transition for a subject in State #1 at time 4 equals 5, while it equals 7 for a subject in State #2 at time 4. Find the actuarial present value at time 3 of future cash flows for a subject in State #2 at time 3.

Question 6 solution

You can compute $APV_{2@3}$, the actuarial present value of these cash flows as seen from State #2 at time 3, by splitting off the first time period from the remaining periods:

$$APV_{2@3} = Q_3^{(2,1)}\,{}_4C^{(2,1)}v + Q_3^{(2,1)}vAPV_{1@4} + Q_3^{(2,2)}\,{}_4C^{(2,2)}v + Q_3^{(2,2)}vAPV_{2@4},$$

which equals $(0.4)(4)(0.8) + (0.4)(0.8)(5) + (0.6)(5)(0.8) + (0.6)(0.8)(7) = 8.64$.

7. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1}\begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Driver C is now Standard at the start of the fourth year. For $k = 0, 1$ there is a cost of 5 at time $3 + k$ if Driver C is Standard at the start of year $3 + k + 1$. Find the actuarial present value now of these costs for Driver C using 15% interest.

Question 7 solution

The triple-product summation for this actuarial present value is

$$(1)(5)(1) + Q_3^{(2,2)}(5)v = 5 + \frac{(0.65)(5)}{1.15} = 7.8261.$$

8. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Driver F is Standard now, at the start of the fourth year. For $k = 0, 1$, there is a cost of $10(1.1)^k$ at the end of year $4+k$ for a transition from Standard at the start of that year to Preferred at the start of the next year. These costs will be funded by allocations ("premiums") P paid at time 3 if Driver F is Standard at time 3 and paid at time 4 if Driver F is Standard at time 4. The allocation is determined to be $P = 3.1879$ by the Equivalence Principle, using 15% interest. Suppose that Driver F is Standard at the start of the fifth year; find the benefit reserve.

Question 8 solution

The benefit reserve is the actuarial present value of future benefits minus that of future benefit premiums, as computed for a driver in State #2 at time 4. Since there is only one year of possible benefits and one certain premium, the benefit reserve is

$$Q_4^{(2,1)}[10(1.1)](\frac{1}{1.15}) - 3.1879 = \frac{(0.36)(11)}{1.15} - 3.1879 = 0.25558.$$

9. The status of residents in a Continuing Care Retirement Community (CCRC) is modeled by a non-homogeneous Markov Chain with three states: Independent Living (#1), Health Center (#2), and Gone (#3). The transition-probability matrices for a new entrant (time 0) are

$$Q_0 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A new entrant, Resident G, enters Independent Living at time 0. The CCRC undergoes a cost of 100 at the end of year k for transition from Independent Living at the start of that year to Health Center at the start of the next year, for all k. The CCRC wishes to charge a fee P at the start of each year when Resident G is in Independent Living, with P determined by the Equivalence Principle to be $P = 17.97$ using 25% interest. Suppose that Resident G is in Independent Living at the start of the third year. Find the benefit reserve.

Question 9 solution

The benefit reserve is the actuarial present value of future benefits minus that of future benefit premiums, as computed for a resident in State #1 at time 2. Since there is only one year of possible benefits and one certain and one possible premium, the benefit reserve is

$$Q_2^{(1,2)}(100)v - [17.97 + Q_2^{(1,1)}(17.97)v] = -6.28.$$

10. The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices $\boldsymbol{Q}_n$ from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$\boldsymbol{Q}_n = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} + \frac{1}{n+1} \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Driver D is Standard now, at the start of the fourth year. For $k = 0, 1$, there is a cost of $10(1.1)^k$ at the end of year $4+k$ for a transition from Standard at the start of that year to Preferred at the start of the next year. These costs will be funded by allocations ("premiums") P paid at time 3 if Driver D is Standard at time 3 and paid at time 4 if Driver D is Standard at time 4. The allocation is determined by the Equivalence Principle, using 15% interest. Find P.

Question 10 solution

The benefit-premium can be computed as the actuarial present value of the benefits divided by the actuarial present value of premiums of 1. The triple-product summation for the actuarial present value of the benefits is $Q_3^{(2,1)}(10)(\frac{1}{1.15})+Q_3^{(2,2)}Q_4^{(2,1)}[10(1.1)](\frac{1}{1.15^2}) = \frac{(0.35)(10)}{1.15} + \frac{(0.65)(0.36)[10(1.1)]}{1.15^2} = 4.9898$. That for the actuarial present value of premiums of 1 is

$$(1)(1)(1) + Q_3^{(2,2)}(1)v = 1 + \frac{(0.65)(1)}{1.15} = 1.5652.$$

Thus the benefit premium is $\frac{4.9898}{1.5652} = 3.1879$.

11. The status of residents in a Continuing Care Retirement Community (CCRC) is modeled by a non-homogeneous Markov Chain with three states: Independent Living (#1), Health Center (#2), and Gone (#3). The transition-probability matrices for a new entrant (time 0) are

$$\boldsymbol{Q}_0 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{Q}_1 = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{Q}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A new entrant, Resident E, enters Independent Living at time 0. The CCRC undergoes a cost of 100 at the end of year k for transition from Independent Living at the start of that year to Health Center at the start of the next year, for all k. The CCRC wishes to charge a fee P at the start of each year when Resident E is in Independent Living, with P determined by the Equivalence Principle using 25% interest. Find P.

Question 11 solution

The benefit-premium can be computed as the actuarial present value of the benefits divided by the actuarial present value of premiums of 1. The triple-product summation for the actuarial present value of the benefits is

$$Q_0^{(2,2)}(100)v + Q_0^{(1,1)}Q_1^{(1,2)}(100)v^2 + {}_2Q_0^{(1,1)}Q_2^{(1,2)}(100)v^3,$$

which equals $(0.2)(100)(0.8)+(0.7)(0.3)(100)(0.64)+(0.35)(0.2)(100)(0.512) = 33.024$; here ${}_2Q_0^{(1,1)}$ was computed as the $(1,1)$-entry of $\boldsymbol{Q}_0\boldsymbol{Q}_1$, namely 0.35. The actuarial present value of premiums of 1 is

$$1 + Q_0^{(1,1)}v + {}_2Q_0^{(1,1)}v^2 + {}_3Q_0^{(1,1)}v^3,$$

which is $1+(0.7)(0.8)+(0.35)(0.64)+(0.105)(0.512) = 1.83776$; here ${}_3Q_0^{(1,1)}$ was computed as the $(1,1)$-entry of $\boldsymbol{Q}_0\boldsymbol{Q}_1\boldsymbol{Q}_2$, namely 0.105. Finally, the benefit premium is $\frac{33.024}{1.83776} = 17.970$.