The actuarial CTE risk measure for heavy-tailed losses

A new estimator and confidence intervals

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Based on joint work with

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CTE

$$\mathbb{C}(t) = \mathbf{E}\left(X \middle| X > Q(t)\right)$$

$$= \frac{1}{1-t}\int_t^1 Q(s)ds$$

Estimator

$$\widehat{\mathbb{C}}_n(t) = \frac{1}{1-t}\int_t^1 Q_n(s)ds,$$

where

$$Q_n(s) = X_{i:n} \quad \forall\ s \in \left(\frac{i-1}{n}, \frac{i}{n}\right]$$

Theorem 1

- Vytaras Brazauskas
- Bruce Jones
- Madan Puri
- RZ

$\mathbf{E}[X^2] < \infty$

For every $t \in (0,1)$,

$$\sqrt{n}\Big(\widehat{\mathbb{C}}_n(t) - \mathbb{C}(t)\Big)(1-t) \to_d \mathcal{N}(0, \sigma^2(t))$$

with

$$\sigma^2(t) = \int_t^1 \int_t^1 (\min(x,y) - xy)\, dQ(x) dQ(y)\,.$$

$\mathbf{E}[X^2] = \infty$

F is regularly varying at infinity:

$$\lim_{x \to \infty} \frac{1 - F(tx)}{1 - F(x)} = \frac{1}{t^{1/\gamma}} \qquad \forall t > 0$$

E.g.: Pareto, gen. Pareto, Burr, Fréchet, Student, etc.

Pareto

$$1 - F(x) = \left(\frac{x_0}{x} \right)^{1/\gamma}, \quad x \geq x_0,$$

$\gamma \in (0, 1/2)$	$\gamma \in (1/2, 1)$	$\gamma \in (1, \infty)$
$\mathbf{E}[X^2] < \infty$	$\mathbf{E}[X^2] = \infty$	$\mathbf{E}[X] = \infty$
	$\mathbf{E}[X] < \infty$	

Write

$$\mathbb{C}(t) = \frac{1}{1-t}\int_t^{1-k/n} Q(s)ds$$

$$+ \frac{1}{1-t}\int_{1-k/n}^{1} Q(s)ds$$

$$\approx \frac{1}{1-t}\int_t^{1-k/n} Q_n(s)ds$$

$$+ \frac{1}{1-t}\int_{1-k/n}^{1} \widetilde{Q}(s)ds$$

where $\widetilde{Q}(s)$ is Weissman's estimator

- Weissman's estimator

$$\widetilde{Q}(s) = \frac{(k/n)^{\widehat{\gamma}} X_{n-k:n}}{(1-s)^{\widehat{\gamma}}}$$

- Integers $k \in \{1, \ldots, n\}$

$$k \to \infty \qquad \text{and} \qquad k/n \to 0$$

- Hill's estimator

$$\widehat{\gamma}_n = \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1:n} - \log X_{n-k:n}$$

CTE estimator:

$$\boxed{\widetilde{\mathbb{C}}_n(t) = \frac{1}{1-t} \int_t^{1-k/n} Q_n(s) ds + \frac{k X_{n-k:n}}{n\,(1-t)\,(1-\widehat{\gamma})}}$$

Theorem 2

- Abdelhakim Necir
- Abdelaziz Rassoul
- RZ

$\gamma \in (1/2, 1)$ [...plus technical assumptions]

For every $t \in (0, 1)$,

$$\boxed{\frac{\sqrt{n}\Big(\widetilde{\mathbb{C}}_n(t) - \mathbb{C}(t)\Big)(1-t)}{(k/n)^{1/2} X_{n-k:n}} \to_d \mathcal{N}\left(0, \sigma_\gamma^2\right)}$$

where

$$\sigma_\gamma^2 = \frac{\gamma^4}{(1-\gamma)^4(2\gamma-1)}$$

$\varsigma \in (0,1)$ significance level

$z_{\varsigma/2}$ $(1-\varsigma/2)$-quantile of $\mathcal{N}(0,1)$

$X_1, \dots, X_n$ i.i.d. $RV_\gamma(\infty)$

Choose 'optimal' k (many papers)

Confidence interval for CTE:

$$\boxed{\widetilde{\mathbb{C}}_n(t) \pm z_{\varsigma/2} \frac{(k/n)^{1/2} X_{n-k:n}\, \sigma_{\widehat{\gamma}_n}}{(1-t)\sqrt{n}}}$$

where

$$\widetilde{\mathbb{C}}_n(t) = \frac{1}{1-t} \int_t^{1-k/n} Q_n(s) ds + \frac{k X_{n-k:n}}{n\,(1-t)\,(1-\widehat{\gamma})}$$