SOCIETY OF ACTUARIES

Article from:

ARCH 2013.1 Proceedings

August 1- 4, 2012

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Credibility Theory in a Fuzzy Environment

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Presented at the 47th Actuarial Research Conference,
University of Manitoba, Canada
August 2012

ABSTRACT

In insurance, credibility theory (CT) is used to develop a weighted average of the claims experience of an individual contract and the experience for the whole portfolio, where the weight factor is the credibility attached to the individual experience. Recently, Liu and Liu (2002) and Liu (2007), in the study of the behavior of fuzzy phenomena, formulated an alternate version of credibility theory, which involves a weighted average based on the concepts of possibility measure and necessity measure. This latter version of credibility theory will hereafter be referred to as credibility theory in a fuzzy environment (CT-F).

This paper presents an overview of CT-F and discusses its implications.

Keywords: credibility theory, fuzzy theory, fuzzy logic.

1. INTRODUCTION

1.1. Classical Actuarial Credibility

In actuarial science, credibility measures the level of confidence one attaches to a specific data set when it comes to projected risk, for rate-making purposes. Credibility theory aims at efficiently combine information from diverse sources: past and current data, individual risk and collective risk data, etc. In particular, credibility theory is used to develop a weighted average of the claims experience of an individual contract and the experience for the whole portfolio, in order to efficiently project future risk associated with a contract.

The general formula for actuarial credibility takes the linear form

$$C = z R + (1-z) H, \quad 0 \leq z \leq 1 \tag{1}$$

where z expresses the level of credibility contains in information from source R, and (1-z) represents the complement credibility assigned to the alternation data source H.

The key issue is to determine the credibility weight z. Early on, this was accomplished using the limited fluctuations[1] credibility theory (Mowbray, 1914), while currently, the standard approach is to use the greatest accuracy[2] credibility theory based on a Bayesian model and developed by Bühlman (1967) and Bühlman-Straub (1970).

1.2. Concept of Credibility Theory in Fuzzy Environment

Fuzzy logic allows intermediate values to be defined between conventional evaluations like true or false, high or low, tall or short etc. Notions like "rather tall" or "very fast" also can be formulated mathematically and processed by computers, in order to apply a more human−like way of thinking in programming. Fuzzy numbers are numbers that have fuzzy properties. An example of a fuzzy number with triangular membership is provided in Figure 1[3].

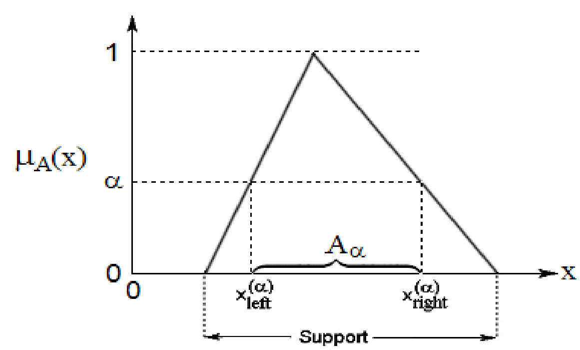

Figure 1: Membership Function for Triangular Fuzzy Number

[1] According to the limited fluctuation credibility (also referred to as American credibility), an insurer's premium should be based solely on its own experience if the experience is significant and stable enough to be considered stable. (Goulet, 1998:8)

[2] The greatest accuracy credibility (also called the European credibility) does not focus on the stability of the experience, but rather on the homogeneity of the experience within the portfolio. (Goulet, 1998:8)

[3] Figure 1 is adapted from Shapiro (2007)

In contrast to the crisp variables of a probability space, fuzzy variables are associated with a possibility space. In their study of the behavior of fuzzy phenomena, Liu and Liu (2002) and Liu (2007) formulated an alternate version of credibility theory, which involves a weighted average based on the concepts of possibility measure and necessity measure.

2. POSSIBILITY SPACE

In a fuzzy environment, the possibility of an event is determined by its most favorable case only, in contrast to the probability of an event, where all favorable cases are accumulated. A brief definition of a possibility space and a summary of its properties follow (Shapiro, 2009).

2.1. Definition.

A possibility space is defined as the 3-tuple (Θ, P(Θ), Pos), where

- $\Theta = (\theta_1, \theta_2, \ldots, \theta_N)$ is a sample space
- P(Θ) also denoted as 2^{Θ}, is the power set of Θ that is the set of all subsets of Θ, and
- Pos is a possibility measure defined on Θ.

2.2. Properties.

The possibility Pos(A) that an event A will occur satisfies the following properties:

- Pos(Θ) = 1
- Pos(Ø) = 0
- $0 \leq$ Pos(A) ≤ 1 for A in P(Θ)
- Pos $\{\cup_i A_i\} = \sup_i$ (Pos $\{A_i\}$) for any collection of $\{A_i\}$ in P(Θ)

3. NECESSITY AND CREDIBILITY MEASURE

3.1. Necessity Measure of a set

Let A be a set on a possibility space (Θ, P(Θ), Pos), then the necessity measure Nec{A} of A is defined as the impossibility of the complement set A^c, and is given by (Zadeh, 1978)

$$Nec\{A\} = 1 - Pos\{A^C\} \qquad \text{for any event A} \qquad (2)$$

3.2. Credibility Measure in a fuzzy environment (CT-F)

Liu and Liu (2003) defined the credibility in a fuzzy environment as the average[4] of the possibility and necessity measures

$$Cr\{A\} = \frac{1}{2}(Pos\{A\} + Nec\{A\}) \qquad \text{for any event A} \qquad (3)$$

To better conceptualize these measures, we consider a triangular fuzzy variable for which we compute the possibility, necessity, and credibility measures.

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[4] This weight of course is a major difference between the standard credibility theory (CT) and the credibility theory in a fuzzy environment (CT-F). In the CT, the main task is to find the weight z. However, in CT-F a choice of 0.5 is preliminary made.

4. POSSIBILITY, NECESSITY AND CREDIBILITY OF A FUZZY EVENT

Let X be a triangular fuzzy number (TFN) on a possibility space (Θ, P(Θ), Pos). Let $\mu_X(x)$ be its membership function, where μ and r are real numbers. The explicit form of the membership function for a TF variable is provided (Zadeh, 1965).

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ \dfrac{c-x}{c-b} & \text{if } b \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

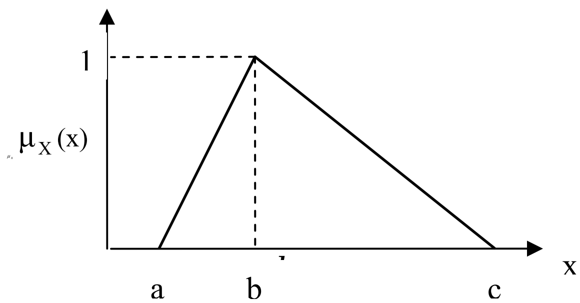

Figure 2: TF number

Then, the possibility, necessity, and credibility of this fuzzy event are defined as follows.

- Figure 3 shows a representation of the possibility of a fuzzy event[5] $\{X \le x_0\}$.

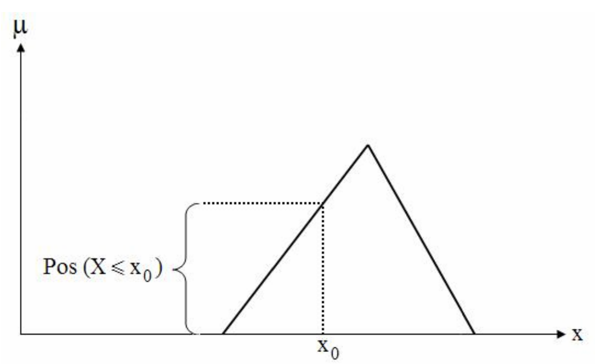

Figure 3: Possibility of a triangular fuzzy event

The explicit expression for the possibility of a TFN is given by

$$\text{Pos}\{X \le x_0\} = \sup_{x \le x_0} \mu(x) = \begin{cases} 0 & x_0 \le a \\ \dfrac{x_0 - a}{b-a} & a \le x_0 \le b \\ 1 & x_0 \ge b \end{cases} \quad (4)$$

[5] Figure 2 is adapted from Tanaka and Guo (1999)

- Figure 4 shows a representation of the necessity of a fuzzy event $\{X \le x_0\}$:

$$\text{Nec}\{X \le x_0\} = 1 - \text{Pos}\{X > x_0\} = 1 - \sup_{x > x_0} \mu(x)$$

$$= 1 - \begin{cases} 1 \\ \dfrac{c - x_0}{c - b} \\ 0 \end{cases} \tag{5}$$

$$= \begin{cases} 0 & x_0 \le b \\ \dfrac{x_0 - b}{c - b} & b \le x_0 \le c \\ 1 & x_0 \ge c \end{cases}$$

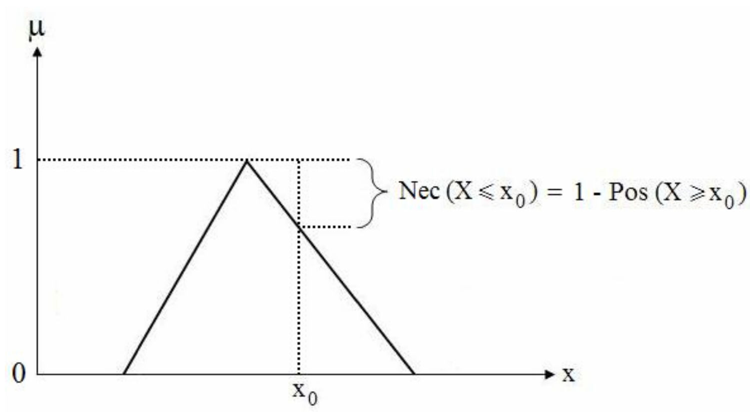

Figure 4: Necessity of a triangular fuzzy event

- The credibility of a fuzzy event is then obtained as the average of its possibility and necessity.

$$\text{Cr}\{X \le x_0\} = \frac{1}{2}\left(\text{Pos}\{X \le x_0\} + \text{Nec}\{X \le x_0\}\right) \tag{6}$$

5. MOMENTS AND RELATED FUNCTIONS

Credibility inversion theorem.
Let ξ be a TFN on a possibility space (Θ, P(Θ), Pos), and let B be a Borel set of $\Re$. Then the CT-F have the following form (Liu, 2007)

$$\text{Cr}\{\xi \in \text{B}\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^C} \mu(x)\right) \tag{7}$$

Corollaries:

$$\mathrm{Cr}\{\xi = x\} = \frac{1}{2}\left(\mu(x) + 1 - \sup_{x \neq y}\mu(y)\right), \quad \forall x \in \Re \tag{8}$$

$$\mu(x) = \left(2\,\mathrm{Cr}\{\xi = x\}\right) \wedge 1, \quad x \in \Re \tag{9}$$

$$\mathrm{Cr}\{\xi \le x\} = \frac{1}{2}\left(\sup_{y \le x}\mu(y) + 1 - \sup_{y > x}\mu(y)\right), \quad \forall x \in \Re \tag{10}$$

Expected Value Operator

The expected value is defined by:

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \ge x\}dx - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \le x\}dx,$$

provided that at least one of these integrals is finite.

6. CONCLUSION

Although they conceptually differ, actuarial credibility and credibility in fuzzy environment have some similar features. They both have a dual factor (a set A and its complement set A^c), for example.

Our ongoing study will explore these similarities as well as the rational for the choice of ½ in the CT-F formula. In particular, we will investigate how CT-F can help improve the Bühlmann credibility model.

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