Optimal Reinsurance with Positive Dependence

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Classical Optimal Reinsurance Problem

Statement of the Optimization Problem

$$\inf_{I\in\mathcal{D}} \rho(I(X)) \text{ subject to } \pi(R(X)) = p.$$

- $R(X)$ – ceded risk, $I(X) = X - R(X)$ – retained risk;
- Find a strategy to minimize the retained risk $I(X)$.

Ingredients of Optimization Problem

- π – the premium principle for reinsurance;
- ρ – risk measure as optimization criterion;
- $\mathcal{D}$ – admissible strategy class.

Traditional risk measure: $\rho(X) = \mathbb{E}[u(X)]$

$u(x)$ is a convex function. For example

- $u(x) = x^2$ – minimize variance;
- $u(x) = e^{\gamma x}$ – maximize utility of insurer's wealth:
- $u(x) = (x - \mathbb{E}[X])_+^2$ – minimize semi-variance.

Mean-Variance Premium Principle: $\mathbb{E}[X] = g(\pi(X), \mathbb{D}X)$

- Expected value premium: $\pi(X) = (1+\theta)\mathbb{E}[X]$;
- Variance premium: $\pi(X) = \mathbb{E}[X] + \beta \mathrm{Var} X$;
- Standard deviation premium: $\pi(X) = \mathbb{E}[X] + \beta \mathbb{D} X$.

Solutions to Classical Optimizaition Problems

Target – minimizing variance.

- Pure variance premium: quota reinsurance – $R(x) = \alpha\, x$
- Expectation premium: stop loss reinsurance - $R(x) = (x - d)_+$
- Mean-Variance premium: change loss reinsurance – $R(x) = \alpha\,(x - d)_+$.

Reference: Borch 1969, Kaluszka 2001.

Generalization of Classical Model

- Different premium principles, risk measures;
- Consider multiple risk instead of one-dimensional risk.

Motivation

A Practical Problem

Consider an auto insurance policy covering two source of loss: vehicle damage and personal injury. Usually, different types of loss have to be reinsured separately.
—How to make an optimal reinsurance arrangement?

Modeling

- The risk is modeled by (X_1, X_2), $X_1, X_2 \geq 0$.
- The reinsurance strategy (I_1, I_2) is applied, i.e.
  For each X_i, the insurer retains $I_i(X_i)$.
- Objective: Minimize the total retained risk $I_1(X_1) + I_2(X_2)$.

Notations

Statement of the Optimization Problem

$$\inf_{(I_1,I_2)\in\mathcal{D}} \mathbb{E}\left[u(I_1(X_1)+I_2(X_2))\right] \text{ subject to } \mathbb{E}\left[(I_1(X_1)+I_2(X_2))\right]=p.$$

- Expectation premium principle
- Convex risk measure: $\rho(X)=\mathbb{E}\left[u(X)\right]$ with convex u.

Admissible Strategy Classes

$$\begin{aligned}
\mathcal{D} &= \left\{(I_1,I_2)\;\middle|\; \begin{array}{l} I_i(x) \text{ is non-decreasing in } x\geq 0 \text{ satisfying} \\ 0\leq I_i(x)\leq x \text{ for } i=1,2. \end{array}\right\};\\
\mathcal{D}^p &= \left\{(I_1,I_2)\in\mathcal{D}\,|\mathbb{E}\left[(I_1(X_1)+I_2(X_2))\right]=p\right\};\\
\mathcal{D}^p_{sl} &= \left\{(I^{d_1},I^{d_2})\in\mathcal{D}^p \mid I^{d_i}(x)=x\wedge d_i,\;\; i=1,2\right\}
\end{aligned}$$

$\mathcal{D}^p$ - global strategy class; $\mathcal{D}^p_{sl}$ - (bivariate) stop-loss strategy class.

Comments on the Bivariate Model

Individualized Strategy vs Global Strategy

- Global strategy $I(X_1, X_2) = I(X_1 + X_2) \longrightarrow$ classical problem;
- Individualized Strategy $I(X_1, X_2) = I_1(X_1) + I_2(X_2)$.

Independent Case

Heerwaarden et al (1989) has shown that: if X_1 and X_2 are independent, the optimal strategy has the stop loss form, i.e. $(I_1(x_1), I_2(x_2)) = (x_1 \wedge d_1, x_2 \wedge d_2)$.

Ideas to Solve the Problem

Under certain dependence structure,

- Show the optimality of bivariate stop loss strategy;
- Find out optimal solution among the stop loss strategy.

Dependence Structure

Definition: Stochastically Increasing

X is stochastically increasing in Y, denoted as $X \uparrow_{SI} Y$, if $\mathbb{P}\{X > x|Y = y_1\} \leq \mathbb{P}\{X > x|Y = y_2\}$, for any $x, y_1 \leq y_2$; or equivalently, if $X|\{Y = y_1\} \leq_{st} X|\{Y = y_2\}$ for any $y_1 \leq y_2$.

Examples

- Independent or comonotonic random variables;
- Common shock: $X_1 = Y_1 \times Z, X_2 = Y_2 \times Z$.
- Random variables linked by typical copulas: such as Gaussian/Gumbel/Clayton copula with coefficient restriction.

Optimization — $\mathcal{D}^p$ vs $\mathcal{D}^p_{sl}$

Theorem 1 - Equivalence of Minimization in $\mathcal{D}^p$ and $\mathcal{D}^p_{sl}$

If $X_1 \uparrow_{SI} X_2$ and $X_2 \uparrow_{SI} X_1$, then for any $(I_1, I_2) \in D^p$, there exists $(I^{d_1}, I^{d_2}) \in D^p_{sl}$ such that

$$\mathbb{E}\left[u(I^{d_1}(X_1) + I^{d_2}(X_2)\right) \leq \mathbb{E}\left[u(I_1(X_1) + I_2(X_2))\right],$$

for any convex function $u(x)$.

Application in Dynamic Model

Consider a compound Poisson model:

$$U^{(I_1,I_2)}(t) = u + p\,t - \sum_{i=1}^{N(t)}(I_1(X_{1,i}) + I_2(X_{2,i})),$$

where $(X_{1,i}, X_{2,i}) \sim_{i.i.d.} (X_1, X_2)$. Denote by $\phi^{(I_1,I_2)}(u)$ the ruin probability of the surplus process $U^{(I_1,I_2)}(t)$, then there exists $(d_1, d_2) \in \mathcal{D}^p_{sl}$ such that $\phi^{(d_1,d_2)}(u) \leq \phi^{(I_1,I_2)}(u)$.

The Premium Constraint

The Curve Determined by Premium Constraint

$$L = \left\{ (d_1, d_2) \,\middle|\, \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p, d_1, d_2 \geq 0 \right\}$$

Properties of the Curve L

- On L, $d_2 = L(d_1)$ is a one-to-one mapping;
- $L(d_1)$ a convex function, with $\frac{\partial d_2}{\partial d_1} = -\frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)}$;
- Denote the endpoints of L by $(\underline{d}_1, \overline{d}_2)$ and $(\underline{d}_2, \overline{d}_1)$. For simplicity, assume $\overline{d}_1 = \overline{d}_2 = \infty$.

Graph of L-Curve

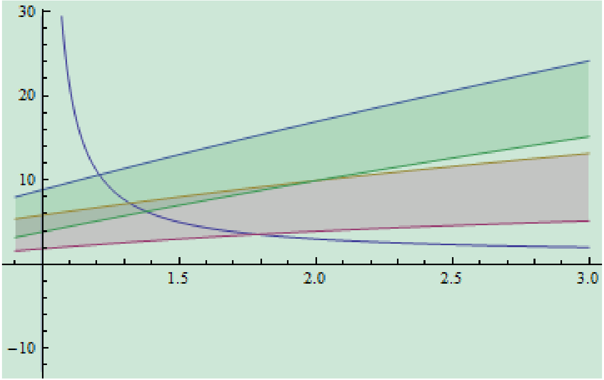

Figure: L-Curve and solution area

Two Optimization Problems

Problem Description

$$\inf_{(d_1,d_2)\in\mathcal{D}_{sl}} Var[I^{d_1}(X_1) + I^{d_2}(X_2)] \quad (1)$$

$$\inf_{(d_1,d_2)\in\mathcal{D}_{sl}} \mathbb{E}\left[\exp\{s\,(I^{d_1}(X_1) + I^{d_2}(X_2))\}\right], s \in \mathbb{R}\,. \quad (2)$$

Explicit Solutions

The solutions to (1) and (2) exist and are determined by:

$$\begin{cases} \mathbb{E}\left[(X_2 - d_2)_- | X_1 > d_1\right] = \mathbb{E}\left[(X_1 - d_1)_- | X_2 > d_2\right] \\ \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p. \end{cases}$$

$$\begin{cases} \mathbb{E}\left[\exp\{s\,(X_2 - d_2)_-\} | X_1 > d_1\right] = \mathbb{E}\left[\exp\{s\,(X_1 - d_1)_-\} | X_2 > d_2\right] \\ \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p. \end{cases}$$

Multivariate Dependence Structure

Definition: Positive Dependent through Stochastic Ordering

Random vector $\mathbf{X}$ is said to be *stochastically increasing* in random variable Y, denoted as $\mathbf{X} \uparrow_{SI} Y$, if $\mathbf{X}|Y = y_1 \leq_{st} \mathbf{X}|Y = y_2$ for any $y_1 \leq y_2$;
Random vector $\mathbf{X}$ is said to be *positive dependent through stochastic ordering* (PDS), if $(X_i, i \neq j) \uparrow_{SI} X_j$ for all $j = 1, 2, \cdots, n$.

Examples of Stochastically Increasing

If $\mathbf{X}$ is linked by one of the following copulas, then $\mathbf{X}$ is PDS:

- The multivariate independence/comonotonicity copula;
- The multivariate Gaussian copula with nonnegative correlation matrix.

Optimality of Stop Loss Strategy

Multivariate Strategy Classes

$$\mathcal{M} = \left\{ \mathbf{I} \;\middle|\; \begin{array}{l} I_i(x) \text{ is non-decreasing in } x \geq 0 \text{ satisfying} \\ 0 \leq I_i(x) \leq x \text{ for } i = 1, 2, \cdots, n \end{array} \right\},$$

$$\mathcal{M}^p = \left\{ \mathbf{I} \in \mathcal{M} \;\middle|\; \sum_{i=1}^{n} \mathbb{E}\left[I_i(X_i)\right] = p \right\}; \mathcal{M}^p_{sl} = \{\mathbf{I} \in \mathcal{M}^p | I_i(x) = x \wedge d_i\}$$

Theorem 2 - Generalization of Theorem 1

If $\mathbf{X}$ is PDS, then for any convex function $u(x)$,

$$\inf_{\mathbf{I} \in \mathcal{M}^p_{sl}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right] = \inf_{\mathbf{I} \in \mathcal{M}^p} \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right]$$

Unbinding Constraint and Admissible Strategy Class

Binding and Unbinding Constraints

Assume expected value premium principle:

$$\begin{cases} \text{Binding:} \;\; \pi\left(\sum_{i=1}^{n} I_i(X_i)\right) = p_2 \iff \mathbb{E}\left[\sum_{i=1}^{n} I_i(X_i)\right] = p, \\ \text{Unbinding:} \;\; \pi\left(\sum_{i=1}^{n} I_i(X_i)\right) \leq p_2 \iff \mathbb{E}\left[\sum_{i=1}^{n} I_i(X_i)\right] \geq p. \end{cases}$$

Admissible Strategy Classes

$$\mathcal{M}^{\geq p} = \left\{ \mathbf{I} \in \mathcal{M} \,\middle|\, \sum_{i=1}^{n} \mathbb{E}\left[I_i(X_i)\right] \geq p \right\}; \mathcal{M}_{sl}^{\geq p} = \{\mathbf{I} \in \mathcal{M}^p | I_i(x) = x \wedge d_i\}.$$

Clearly, $\mathcal{M}^p \subset \mathcal{M}^{\geq p}$ and $\mathcal{M}_{sl}^p \subset \mathcal{M}_{sl}^{\geq p}$

Equivalence of Binding and Unbinding Constraints

Proposition 3 - Optimization in $\mathcal{M}^{\geq p}$ vs $\mathcal{M}_{sl}^{p}$

If **X** is PDS, then for any increasing convex function $u(x)$,

$$\inf_{\mathbf{I}\in\mathcal{M}_{sl}^{p}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right] = \inf_{\mathbf{I}\in\mathcal{M}^{\geq p}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right]$$

Intuition

The insurer tend to exhaust all the premium budget to cede as much risk as possible. As an extreme case, assume the premium budget is sufficiently large, the insurer would choose to cede all the risk to the reinsurer and completely avoid the risk.

Release the Premium Constraint

Two Larger Strategy Classes - $\mathcal{M}^{\geq p}$ and $\mathcal{M}_{sl}^{\geq p}$

$$\mathcal{M}^{\geq p} = \left\{ \mathbf{I} \in \mathcal{M} \,\middle|\, \sum_{i=1}^{n} \mathbb{E}\left[I_i(X_i) \right] \geq p \right\},$$

$$\mathcal{M}_{sl}^{\geq p} = \left\{ \mathbf{I} \in \mathcal{M}^{\geq p} \,|\, I_i(x) = x \wedge d_i \right\}.$$

Interpretation: there is a budget limit for reinsurance premium.

Proposition 3 - Optimization in $\mathcal{M}^{\geq p}$

If $\mathbf{X}$ is PDS, then for any increasing convex function $u(x)$,

$$\inf_{\mathbf{I} \in \mathcal{M}_{sl}^{p}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i) \right) \right] = \inf_{\mathbf{I} \in \mathcal{M}^{\geq p}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i) \right) \right]$$

Conclusive Remarks

Conclusion

- Under PDS dependence, optimal reinsurance strategy is multivariate stop loss form;
- Explicit solutions could be derived in certain bivariate model;
- Binding and unbinding constraints are equivalent.

Future Work

- Continue studying the multivariate model.
- Consider more general premium principles.

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Thank You !