

# Quantitative Finance and Investment Core Formula Sheet

Spring 2018

Morning and afternoon exam booklets will include a formula package identical to the one attached to this study note. The exam committee believe that by providing many key formulas, candidates will be able to focus more of their exam preparation time on the application of the formulas and concepts to demonstrate their understanding of the syllabus material and less time on the memorization of the formulas.

The formula sheet was developed sequentially by reviewing the syllabus material for each major syllabus topic. Candidates should be able to follow the flow of the formula package easily. We recommend that candidates use the formula package concurrently with the syllabus material. Not every formula in the syllabus is in the formula package. **Candidates are responsible for all formulas on the syllabus, including those not on the formula sheet.**

Candidates should carefully observe the sometimes subtle differences in formulas and their application to slightly different situations. For example, there are several versions of the Black-Scholes-Merton option pricing formula to differentiate between instruments paying dividends, tied to an index, etc. Candidates will be expected to recognize the correct formula to apply in a specific situation of an exam question.

Candidates will note that the formula package does not generally provide names or definitions of the formula or symbols used in the formula. With the wide variety of references and authors of the syllabus, candidates should recognize that the letter conventions and use of symbols may vary from one part of the syllabus to another and thus from one formula to another.

We trust that you will find the inclusion of the formula package to be a valuable study aide that will allow for more of your preparation time to be spent on mastering the learning objectives and learning outcomes.

In sources where some equations are numbered and others are not ( $nm$ ) denotes that there is no number assigned to that particular equation.

## Chapter 2

$$(2.10) \quad \begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r\Delta) & (1+r\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$(2.66) \quad S_t = \frac{1}{1+r} [\mathbb{Q}_{up}(S_t + \sigma\sqrt{\Delta}) + \mathbb{Q}_{down}(S_t - \sigma\sqrt{\Delta})]$$

$$(2.67) \quad C_t = \frac{1}{(1+r)} [\mathbb{Q}_{up}C_{t+\Delta}^{up} + \mathbb{Q}_{down}C_{t+\Delta}^{down}]$$

$$(2.70) \quad S = \frac{1+d}{1+r} [S^u\mathbb{Q}_{up} + S^d\mathbb{Q}_{down}]$$

$$(2.71) \quad C = \frac{1}{1+r} [C^u\mathbb{Q}_{up} + C^d\mathbb{Q}_{down}]$$

(page 26)  $\mathbb{E}^{\mathbb{Q}} \left[ \frac{C_{t+\Delta}}{C_t} \right] \approx 1 + r\Delta$

## Chapter 3

$$(3.37) \quad \sum_{i=1}^n f\left(\frac{t_i + t_{i-1}}{2}\right) (t_i - t_{i-1}) \rightarrow \int_0^T f(s) ds$$

$$(3.49) \quad \int_0^T g(s) df(s) \approx \sum_{i=1}^n g\left(\frac{t_i + t_{i-1}}{2}\right) (f(t_i) - f(t_{i-1}))$$

## Chapter 4

$$(4.24) \quad dF(t) = F_s dS_t + F_r dr_t + F_t dt + \frac{1}{2} F_{ss} dS_t^2 + \frac{1}{2} F_{rr} dr_t^2 + F_{sr} dS_t dr_t$$

## Chapter 5

$$(5.11) \quad \mathbb{E}[S_t | I_u] = \int_{-\infty}^{\infty} S_t f(S_t | I_u) dS_t, \quad u < t$$

$$(5.18) \quad P(\Delta F(t) = +a\sqrt{\Delta}) = p$$

$$(5.19) \quad P(\Delta F(t) = -a\sqrt{\Delta}) = 1 - p$$

$$(5.44) \quad P(X_{t+s} \leq x_{t+s} | x_t, \dots, x_1) = P(X_{t+s} \leq x_{t+s} | x_t)$$

$$(5.45) \quad r_{t+\Delta} - r_t = \mathbb{E}[(r_{t+\Delta} - r_t) | I_t] + \sigma(I_t, t) \Delta W_t$$

$$(5.48) \quad dr_t = \mu(r_t, t) dt + \sigma(r_t, t) dW_t$$

$$(5.49) \quad \begin{bmatrix} r_{t+\Delta} \\ R_{t+\Delta} \end{bmatrix} = \begin{bmatrix} \alpha_1 r_t + \beta_1 R_t \\ \alpha_2 r_t + \beta_2 R_t \end{bmatrix} + \begin{bmatrix} \sigma_1 W_{t+\Delta}^1 \\ \sigma_2 W_{t+\Delta}^2 \end{bmatrix}$$

## Chapter 6

$$(6.3) \quad \mathbb{E}_t[S_T] = \mathbb{E}[S_T | I_t], \quad t < T$$

$$(6.4) \quad \mathbb{E}|S_t| < \infty$$

$$(6.5) \quad \mathbb{E}_t[S_T] = S_t, \text{ for all } t < T$$

$$(6.9) \quad \mathbb{E}_t^{\mathbb{Q}}[e^{-ru} B_{t+u}] = B_t, \quad 0 < u < T - t$$

$$(6.10) \quad \mathbb{E}_t^{\mathbb{Q}}[e^{-ru} S_{t+u}] = S_t, \quad 0 < u$$

$$(6.21) \quad \Delta M_t = \Delta N_t^G - \Delta N_t^B$$

$$(6.26) \quad P(\Delta N_t^G = 1) \approx \lambda^G \Delta > P(\Delta N_t^B = 1) \approx \lambda^B \Delta$$

$$(6.27) \quad \mathbb{E}[\Delta M_t] \approx \lambda^G \Delta - \lambda^B \Delta > 0$$

$$(6.44) \quad \Delta X_t \sim N(\mu \Delta, \sigma^2 \Delta)$$

$$(6.46) \quad X_{t+T} = X_0 + \int_0^{t+T} dX_u$$

$$(6.50) \quad Z_t = X_t - \mu t$$

$$(6.54) \quad \mathbb{E}_t[Z_{t+T}] = X_t - \mu t$$

$$(6.55) \quad \mathbb{E}_t[Z_{t+T}] = Z_t$$

$$(6.56) \quad \Delta S_t \sim N(0, \sigma^2 \Delta)$$

$$(6.57) \quad S_0 = 0$$

$$(6.58) \quad Z_t = S_t^2$$

$$(6.60) \quad \mathbb{E}_t[Z_{t+T} - \sigma^2(T+t)] = Z_t - \sigma^2 t$$

$$(6.64) \quad I_t \subseteq I_{t+1} \subseteq \dots \subseteq I_{T-1} \subseteq I_T$$

$$(6.65) \quad M_t = \mathbb{E}^{\mathbb{P}}[Y_T | I_t]$$

$$(6.66) \quad \mathbb{E}^{\mathbb{P}}[M_{t+s} | I_t] = M_t$$

$$(6.70) \quad G_T = f(S_T)$$

$$(6.71) \quad B_T = e^{\int_t^T r_s ds}$$

$$(6.72) \quad M_t = \mathbb{E}^{\mathbb{P}} \left[ \frac{G_T}{B_T} \middle| I_t \right]$$

$$(6.106) \quad M_{t_k} = M_{t_0} + \sum_{i=1}^k H_{t_{i-1}} [Z_{t_i} - Z_{t_{i-1}}]$$

$$(6.108) \quad \mathbb{E}_{t_0}[M_{t_k}] = M_{t_0}$$

$$(6.112) \quad C_T = C_t + \int_t^T D_s ds + \int_t^T g(C_s) dM_s$$

## Chapter 7

$$(7.23) \quad \Delta W_k = [S_k - S_{k-1}] - \mathbb{E}_{k-1}[S_k - S_{k-1}]$$

$$(7.26) \quad W_k = \sum_{i=1}^k \Delta W_i$$

$$(7.28) \quad \mathbb{E}_{k-1}[W_k] = W_{k-1}$$

$$(7.29) \quad \mathbb{V}^k = \mathbb{E}_0[\Delta W_k^2]$$

$$(7.30) \quad \mathbb{V} = \mathbb{E}_0 \left[ \sum_{k=1}^n \Delta W_k \right]^2 = \sum_{k=1}^n \mathbb{V}^k$$

$$(7.31) \quad \mathbb{V} > A_1 > 0$$

$$(7.33) \quad \mathbb{V} < A_2 < \infty$$

$$(7.34) \quad \mathbb{V}_{max} = \max_k [\mathbb{V}^k, k = 1, \dots, n]$$

$$(7.35) \quad \frac{\mathbb{V}^k}{\mathbb{V}_{max}} > A_3, \quad 0 < A_3 < 1$$

$$(7.36) \quad \mathbb{E}[\Delta W_k]^2 = \sigma_k^2 h$$

$$(7.56) \quad S_k - S_{k-1} = \mathbb{E}_{k-1}[S_k - S_{k-1}] + \sigma_k \Delta W_k$$

## Chapter 8

$$(8.7) \quad dN_t = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt \end{cases}$$

$$(8.8) \quad M_t = N_t - \lambda t$$

$$(8.9) \quad \mathbb{E}[M_t] = 0$$

$$(8.15) \quad \mathbb{E}(e^{iuX_t}) = \exp(t\phi(u))$$

$$(8.16) \quad \phi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iuy} - 1 - iuy\mathbb{I}_{\{|y|\leq 1\}}) d\nu(y)$$

$$(8.18) \quad X_t = (r - q + \omega)t + Z_t$$

(page 129)  $d\nu(y) = k(y)dy$

$$(8.19) \quad k(y) = \frac{e^{-\lambda_p y}}{\nu y} \mathbb{I}_{y>0} + \frac{e^{-\lambda_n |y|}}{\nu |y|} \mathbb{I}_{y<0}$$

$$(nn) \quad \lambda_p = \left( \frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu} \right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2}$$

$$(nn) \quad \lambda_n = \left( \frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu} \right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2}$$

$$(8.21) \quad \sigma_k \Delta W_k = \begin{cases} \omega_1 & \text{with probability } p_1 \\ \omega_2 & \text{with probability } p_2 \\ \vdots & \vdots \\ \omega_m & \text{with probability } p_m \end{cases}$$

$$(8.22) \quad \mathbb{E}[\sigma_k \Delta W_k]^2 = \sigma_k^2 h$$

$$(8.23) \quad \sum_{i=1}^m p_i \omega_i^2 = \sigma_k^2 h$$

$$(8.28) \quad \omega_i(h) = \bar{\omega}_i h^{r_i}$$

$$(8.29) \quad p_i(h) = \bar{p}_i h^{q_i}$$

$$(8.33) \quad q_i + 2r_i = 1$$

$$(8.34) \quad c_i = \bar{\omega}_i^2 \bar{p}_i$$

$$(8.58) \quad J_t = (N_t - \lambda t)$$

$$(8.60) \quad dS_t = a(S_t, t)dt + \sigma_1(S_t, t)dW_t + \sigma_2(S_t, t)dJ_t$$

$$(8.75) \quad t_0 = 0 < t_1 < \dots < t_n = T$$

$$(8.76) \quad n\Delta = T$$

$$(8.77) \quad S_i = S_{t_i}, \quad i = 0, 1, \dots, n$$

$$(8.78) \quad S_{i+1} = \begin{cases} u_i S_i & \text{with probability } p_i \\ d_i S_i & \text{with probability } 1 - p_i \end{cases}$$

$$(8.79) \quad u_i = e^{\sigma\sqrt{\Delta}}, \quad \text{for all } i$$

$$(8.80) \quad d_i = e^{-\sigma\sqrt{\Delta}}, \quad \text{for all } i$$

$$(8.81) \quad p_i = \frac{1}{2} \left[ 1 + \frac{\mu}{\sigma} \sqrt{\Delta} \right], \quad \text{for all } i$$

$$(8.89) \quad \frac{S_{i+n\Delta}}{S_i}$$

$$(8.91) \quad \log \frac{S_{i+n}}{S_i} = Z \log u + (n - Z) \log d$$

$$(8.92) \quad \log \frac{S_{i+n}}{S_i} = Z \log \frac{u}{d} + n \log d$$

## Chapter 9

$$(9.37) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{k=1}^n \sigma(S_{k-1}, k) [W_k - W_{k-1}] - \int_0^T \sigma(S_u, u) dW_u \right]^2 = 0$$

$$(9.38) \quad S_k - S_{k-1} = a(S_{k-1}, k)h + \sigma(S_{k-1}, k)[W_k - W_{k-1}], \quad k = 1, 2, \dots, n$$

$$(9.39) \quad \mathbb{E} \left[ \int_0^T \sigma(S_t, t)^2 dt \right] < \infty$$

$$(9.41) \quad \sum_{k=1}^n \sigma(S_{k-1}, k) [W_k - W_{k-1}] \rightarrow \int_0^T \sigma(S_t, t) dW_t$$

$$(9.73) \quad \int_0^T x_t dx_t = \frac{1}{2} [x_T^2 - T]$$

$$(9.74) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - T \right]^2 = 0$$

$$(9.76) \quad \text{If } \int_0^T (dx_t)^2 \text{ exists, then } \lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - \int_0^T (dx_t)^2 \right]^2 = 0$$

$$(9.77) \quad \int_0^T dt = T$$

$$(9.78) \quad \int_0^T (dx_t)^2 = \int_0^T dt$$

$$(9.79) \quad (dW_t)^2 = dt$$

$$(9.85) \quad \mathbb{E}_s \left[ \int_0^t \sigma_u dW_u \right] = \int_0^s \sigma_u dW_u, \quad 0 < s < t$$

$$(9.132) \quad \mathbb{E} \left[ \int_0^T f(W_t, t) dW_t \int_0^T g(W_t, t) dW_t \right] = \mathbb{E} \left[ \int_0^T f(W_t, t) g(W_t, t) dt \right]$$

$$(9.133) \quad \mathbb{E} \left[ \int_0^T f(W_t, t) dW_t \right]^2 = \mathbb{E} \left[ \int_0^T f(W_t, t)^2 dt \right]$$

## Chapter 10

(page 170)  $dS_t = a_t dt + \sigma_t dW_t, \quad t \geq 0$

$$(10.36) \quad dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} dt$$

$$(10.37) \quad dF_t = \left[ a_t \frac{\partial F}{\partial S_t} + \frac{\partial F}{\partial t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} \right] dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t$$

$$(10.64) \quad \int_0^t F_s dS_u = [F(S_t, t) - F(S_0, 0)] - \int_0^t \left[ F_u + \frac{1}{2} F_{ss} \sigma_u^2 \right] du$$

$$(10.69) \quad dF = F_t dt + F_{s_1} dS_1 + F_{s_2} dS_2 + \frac{1}{2} [F_{s_1 s_1} dS_1^2 + F_{s_2 s_2} dS_2^2 + 2F_{s_1 s_2} dS_1 dS_2]$$

$$(10.79) \quad Y(t) = \sum_{i=1}^n N_i(t) P_i(t)$$

$$(10.80) \quad dY(t) = \sum_{i=1}^n N_i(t) dP_i(t) + \sum_{i=1}^n dN_i(t) P_i(t) + \sum_{i=1}^n dN_i(t) dP_i(t)$$

$$(10.81) \quad dS_t = a_t dt + \sigma_t dW_t + dJ_t, \quad t \geq 0$$

$$(10.82) \quad E[\Delta J_t] = 0$$

$$(10.83) \quad \Delta J_t = \Delta N_t - \left[ \lambda_t h \left( \sum_{i=1}^k a_i p_i \right) \right]$$

$$(10.84) \quad a_t = \alpha_t + \lambda_t \left( \sum_{i=1}^k a_i p_i \right)$$

$$(10.85) \quad dF(S_t, t) = \left[ F_t + \lambda_t \sum_{i=1}^k (F(S_t + a_i, t) - F(S_t, t)) p_i + \frac{1}{2} F_{ss} \sigma^2 \right] dt + F_s dS_t + dJ_F$$

$$(10.86) \quad dJ_F = [F(S_t, t) - F(S_t^-, t)] - \lambda_t \left[ \sum_{i=1}^k (F(S_t + a_i, t) - F(S_t, t)) p_i \right] dt$$

$$(10.87) \quad S_t^- = \lim_{s \rightarrow t} S_s, \quad s < t$$

## Chapter 11

$$(11.24) \quad dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, \infty)$$

$$(11.30) \quad S_t = S_0 e^{\{(a - \frac{1}{2}\sigma^2)t + \sigma W_t\}}$$

$$(11.34) \quad dS_t = r S_t dt + \sigma S_t dW_t, \quad t \in [0, \infty)$$

$$(11.38) \quad S_T = \left[ S_0 e^{(r - \frac{1}{2}\sigma^2)T} \right] [e^{\sigma W_T}]$$

$$(11.42) \quad Z_t = e^{\sigma W_t}$$

$$(11.50) \quad x_t = \mathbb{E}[Z_t] = e^{\frac{1}{2}\sigma^2 t}$$

$$(11.55) \quad S_t = e^{-r(T-t)} \mathbb{E}_t[S_T]$$

$$(11.72) \quad dS_t = \mu S_t dt + \sigma \sqrt{S_t} dW_t, \quad t \in [0, \infty)$$

$$(11.74) \quad dS_t = \lambda(\mu - S_t)dt + \sigma S_t dW_t$$

$$(11.78) \quad dS_t = -\mu S_t dt + \sigma dW_t$$

$$(11.79) \quad dS_t = \mu dt + \sigma_t dW_{1t}$$

$$(page 191) \quad d\sigma_t = \lambda(\sigma_0 - \sigma_t)dt + \alpha \sigma_t dW_{2t}$$

$$(page 193) \quad dN_t = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt \end{cases}$$

$$(11.83) \quad \frac{dS_t}{S_t} = (\mu - \lambda \kappa)dt + \sigma dW_t + (e^J - 1)dN_t$$

$$(11.84) \quad \frac{dS_t}{S_t} = (\mu - \lambda^* \kappa^*)dt + \sigma d\tilde{W}_t + (e^J - 1)dN_t$$

$$(page 194) \quad S_t = S_0 e^{(r-q+\omega)t + X(t; \sigma, \nu, \theta)}$$

$$(page 194) \quad f(x; \sigma, \nu, \theta) = \int_0^\infty \frac{1}{\sigma \sqrt{2\pi g}} \exp\left(-\frac{(x - \theta g)^2}{2\sigma^2 g}\right) \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu} \Gamma(t/\nu)} dg$$

## Chapter 12

$$(12.3) \quad P_t = \theta_1 F(S_t, t) + \theta_2 S_t$$

$$(12.4) \quad dP_t = \theta_1 dF_t + \theta_2 dS_t$$

$$(12.5) \quad dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t, \quad t \in [0, \infty)$$

$$(12.6) \quad dF_t = F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt + F_s dS_t$$

$$(12.7) \quad dF_t = \left[ F_s a_t + \frac{1}{2} F_{ss} \sigma_t^2 + F_t \right] dt + F_s \sigma_t dW_t$$

$$(12.10) \quad \theta_1 = 1$$

$$(12.11) \quad \theta_2 = -F_s$$

$$(12.12) \quad dP_t = F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt$$

$$(12.16) \quad r(F(S_t, t) - F_s S_t) = F_t + \frac{1}{2} F_{ss} \sigma_t^2$$

$$(12.17) \quad -rF + rF_s S_t + F_t + \frac{1}{2} F_{ss} \sigma_t^2 = 0, \quad 0 \leq S_t, \quad 0 \leq t \leq T$$

$$(12.23) \quad P_t = F(S_t, t) - F_s(S_t, t)S_t$$

$$(12.24) \quad dP_t = dF(S_t, t) - F_s dS_t - S_t dF_s - dF_s(S_t, t)dS_t$$

$$(12.26) \quad dP_t = dF(S_t, t) - F_s dS_t - S_t \left[ \left[ F_{st} + F_{ss} \mu S_t + \frac{1}{2} F_{sss} \sigma_t^2 S_t^2 \right] dt + F_{ss} \sigma_t dW_t \right] - F_{ss} \sigma_t^2 S_t^2 dt$$

$$(12.28) \quad dP_t = dF(S_t, t) - F_s dS_t - S_t [F_{ss}(\mu - r)S_t dt] - F_{ss} \sigma_t^2 S_t^2 dW_t$$

(page 202)  $dW_t^* = (\sigma dW_t + (\mu - r)dt)$

(12.29)  $a_0 F + a_1 F_s S_t + a_2 F_t + a_3 F_{ss} = 0, \quad 0 \leq S_t, \quad 0 \leq t \leq T$

(12.30)  $F(S_T, T) = G(S_T, T)$

(12.32)  $F_t + F_s = 0, \quad 0 \leq S_t, \quad 0 \leq t \leq T$

(12.33)  $F(S_t, t) = \alpha S_t - \alpha t + \beta$

(12.44)  $F(S_t, t) = e^{\alpha S_t - \alpha t}$

(12.46)  $-0.3 F_{ss} + F_{tt} = 0$

(12.47)  $F(S_t, t) = \frac{1}{2} \alpha (S_t - S_0)^2 + \frac{0.3}{2} \alpha (t - t_0)^2 + \beta (t - t_0)$

(12.53)  $F(S_t, t) = -10(S_t - 4)^2 - 3(t - 2)^2, \quad -10 \leq t \leq 10, \quad -10 \leq S_t \leq 10$

## Chapter 13

(13.1)  $a(S_t, t) = \mu S_t$

(13.2)  $\sigma(S_t, t) = \sigma S_t, \quad t \in [0, \infty)$

(13.3)  $-rF + rF_s S_t + F_t + \frac{1}{2} \sigma^2 F_{ss} S_t^2 = 0, \quad 0 \leq S_t, \quad 0 \leq t \leq T$

(13.4)  $F(T) = \max[S_T - K, 0]$

(13.6)  $F(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$

(13.7)  $d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}$

(13.8)  $d_2 = d_1 - \sigma \sqrt{T - t}$

(13.9)  $N(d_1) = \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx$

(13.12)  $a(S_t, t) = \mu S_t$

(13.13)  $\sigma(S_t, t) = \sigma(S_t, t) S_t, \quad t \in [0, \infty)$

(13.14)  $-rF + rF_s S_t + F_t + \frac{1}{2} \sigma(S_t, t)^2 F_{ss} S_t^2 = 0, \quad 0 \leq S_t, \quad 0 \leq t \leq T$

(13.15)  $F(T) = \max[S_T - K, 0]$

(13.34)  $rF - rF_s S_t - \delta - F_t - \frac{1}{2} F_{ss} \sigma_t^2 = 0$

(13.35)  $F(S_{1T}, S_{2T}, T) = \max[0, \max(S_{1T}, S_{2T}) - K]$  (multi-asset option)

(13.36)  $F(S_{1T}, S_{2T}, T) = \max[0, (S_{1T} - S_{2T}) - K]$  (spread call option)

(13.37)  $F(S_{1T}, S_{2T}, T) = \max[0, (\theta_1 S_{1T} + \theta_2 S_{2T}) - K]$  (portfolio call option)

(13.38)  $F(S_{1T}, S_{2T}, T) = \max[0, (S_{1T} - K_1), (S_{2T} - K_2)]$  (dual strike call option)

(13.47)  $\frac{\Delta F}{\Delta t} + rS \frac{\Delta F}{\Delta S} + \frac{1}{2} \sigma^2 S^2 \frac{\Delta^2 F}{\Delta S^2} \approx rF$



$$(13.48) \quad \frac{\Delta F}{\Delta t} \approx \frac{F_{ij} - F_{i,j-1}}{\Delta t}$$

$$(13.49) \quad rS \frac{\Delta F}{\Delta S} \approx rS_j \frac{F_{ij} - F_{i-1,j}}{\Delta S}$$

$$(13.50) \quad rS \frac{\Delta F}{\Delta S} \approx rS_j \frac{F_{i+1,j} - F_{ij}}{\Delta S}$$

$$(13.51) \quad \frac{\Delta^2 F}{\Delta S^2} \approx \left[ \frac{F_{i+1,j} - F_{ij}}{\Delta S} - \frac{F_{ij} - F_{i-1,j}}{\Delta S} \right] \frac{1}{\Delta S}$$

## Chapter 14

$$(14.6) \quad d\mathbb{P}(\bar{z}) = \mathbb{P} \left( \bar{z} - \frac{1}{2} dz_t < z_t < \bar{z} + \frac{1}{2} dz_t \right)$$

$$(14.7) \quad \int_{-\infty}^{\infty} d\mathbb{P}(z_t) = 1$$

$$(14.8) \quad \mathbb{E}[z_t] = \int_{-\infty}^{\infty} z_t d\mathbb{P}(z_t)$$

$$(14.9) \quad \mathbb{E}[z_t - E[z_t]]^2 = \int_{-\infty}^{\infty} [z_t - \mathbb{E}[z_t]]^2 d\mathbb{P}(z_t)$$

$$(14.29) \quad \mathbb{E}_t \left[ \frac{1}{1 + R_t} S_{t+1} \right] = S_t$$

$$(14.31) \quad \mathbb{E}_t^{\mathbb{Q}} \left[ \frac{1}{1 + r_t} S_{t+1} \right] = S_t$$

$$(14.41) \quad z_t \sim N(0, 1)$$

$$(14.42) \quad d\mathbb{P}(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t)^2} dz_t$$

$$(14.43) \quad \xi(z_t) = e^{z_t \mu - \frac{1}{2} \mu^2}$$

$$(14.44) \quad [d\mathbb{P}(z_t)][\xi(z_t)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t^2) + \mu z_t - \frac{1}{2} \mu^2} dz_t$$

$$(14.45) \quad d\mathbb{Q}(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t - \mu)^2} dz_t$$

$$(14.47) \quad d\mathbb{Q}(z_t) = \xi(z_t) d\mathbb{P}(z_t)$$

$$(14.48) \quad \xi(z_t)^{-1} d\mathbb{Q}(z_t) = d\mathbb{P}(z_t)$$

$$(14.53) \quad f(z_{1t}, z_{2t}) = \frac{1}{2\pi\sqrt{|\Omega|}} e^{-\frac{1}{2} [z_{1t} - \mu_1, z_{2t} - \mu_2] \Omega^{-1} \begin{bmatrix} z_{1t} - \mu_1 \\ z_{2t} - \mu_2 \end{bmatrix}}$$

$$(14.54) \quad \Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

$$(14.55) \quad |\Omega| = \sigma_1^2 \sigma_2^2 - \sigma_{12}^2$$

$$(14.56) \quad d\mathbb{P}(z_{1t}, z_{2t}) = f(z_{1t}, z_{2t}) dz_{1t} dz_{2t}$$

$$(14.57) \quad \xi(z_{1t}, z_{2t}) = \exp \left\{ - [z_{1t}, z_{2t}] \Omega^{-1} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \frac{1}{2} [\mu_1, \mu_2] \Omega^{-1} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \right\}$$

$$(14.58) \quad d\mathbb{Q}(z_{1t}, z_{2t}) = \xi(z_{1t}, z_{2t}) d\mathbb{P}(z_{1t}, z_{2t})$$

$$(14.59) \quad d\mathbb{Q}(z_{1t}, z_{2t}) = \frac{1}{2\pi\sqrt{|\Omega|}} e^{-\frac{1}{2} [z_{1t}, z_{2t}] \Omega^{-1} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}} dz_{1t} dz_{2t}$$

$$(14.60) \quad \xi(z_t) = e^{-z_t' \Omega^{-1} \mu + \frac{1}{2} \mu' \Omega^{-1} \mu}$$

$$(14.69) \quad \frac{d\mathbb{Q}(z_t)}{d\mathbb{P}(z_t)} = \xi(z_t)$$

$$(14.74) \quad d\mathbb{Q}(z_t) = \xi(z_t) d\mathbb{P}(z_t)$$

$$(14.75) \quad d\mathbb{P}(z_t) = \xi(z_t)^{-1} d\mathbb{Q}(z_t)$$

$$(14.76) \quad \xi_t = e^{\left( \int_0^t X_u dW_u - \frac{1}{2} \int_0^t X_u^2 du \right)}, \quad t \in [0, T]$$

$$(14.77) \quad \mathbb{E} \left[ e^{\int_0^t X_u^2 du} \right] < \infty, \quad t \in [0, T]$$

$$(14.83) \quad \mathbb{E} \left[ \int_0^t \xi_s X_s dW_s \middle| I_u \right] = \int_0^u \xi_s X_s dW_s$$

$$(14.84) \quad W_t^* = W_t - \int_0^t X_u du, \quad t \in [0, T]$$

$$(14.85) \quad \mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[1_A \xi_T]$$

$$(14.86) \quad dW_t^* = dW_t - X_t dt$$

$$(14.93) \quad d\mathbb{Q} = \xi_T d\mathbb{P}$$

$$(14.122) \quad A_1 \cup A_2 \cup \dots \cup A_n = \Omega$$

$$(14.123) \quad 1_{A_1} + 1_{A_2} + \dots + 1_{A_n} = 1_{\Omega}$$

$$(14.127) \quad \mathbb{E}^{\mathbb{P}}[Z_t 1_{A_i}] = \mathbb{Q}(A_i)$$

$$(14.138) \quad \mathbb{E}^{\mathbb{P}}[g(X_t)] = \int_{\Omega} g(x) f(x) dx$$

(page 248)  $g(X_t) = Z_t h(X_t)$

$$(14.140) \quad \mathbb{E}^{\mathbb{P}}[g(X_t)] = \int_{\Omega} h(x) \tilde{f}(x) dx = \mathbb{E}^{\mathbb{Q}}[h(X_t)]$$

## Chapter 15

$$(15.2) \quad Y_t \sim N(\mu t, \sigma^2 t)$$

$$(15.4) \quad M(\lambda) = \mathbb{E}[e^{Y_t \lambda}]$$

$$(15.10) \quad M(\lambda) = e^{(\lambda \mu t + \frac{1}{2} \sigma^2 \lambda^2 t)}$$

$$(15.15) \quad S_t = S_0 e^{Y_t}, \quad t \in [0, \infty)$$

$$(15.25) \quad \mathbb{E}[S_t | S_u, u < t] = S_u e^{\mu(t-u) + \frac{1}{2} \sigma^2 (t-u)}$$

$$(15.30) \quad Z_t = e^{-rt} S_t$$

$$(15.31) \quad \mathbb{E}^{\mathbb{Q}} [e^{-rt} S_t | S_u, u < t] = e^{-ru} S_u$$

$$(15.32) \quad \mathbb{E}^{\mathbb{Q}} [Z_t | Z_u, u < t] = Z_u$$

$$(15.38) \quad \mathbb{E}^{\mathbb{Q}} [e^{-r(t-u)} S_t | S_u, u < t] = S_u e^{-r(t-u)} e^{\rho(t-u) + \frac{1}{2} \sigma^2 (t-u)} \quad \text{where under } \mathbb{Q}, Y_t \sim N(\rho t, \sigma^2 t)$$

$$(15.39) \quad \rho = r - \frac{1}{2} \sigma^2$$

$$(15.42) \quad \mathbb{E}^{\mathbb{Q}} [e^{-rt} S_t | S_u, u < t] = e^{-ru} S_u$$

$$(15.51) \quad dS_t = r S_t dt + \sigma S_t dW_t^*$$

$$(15.58) \quad C_t = \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T-t)} \max\{S_T - K, 0\}]$$

$$(15.88) \quad dS_t = \mu_t dt + \sigma_t dW_t$$

$$(15.90) \quad d[e^{-rt} S_t] = e^{-rt} [\mu_t - r S_t] dt + e^{-rt} \sigma_t dW_t$$

$$(15.92) \quad dW_t^* = dX_t + dW_t$$

$$(15.97) \quad dX_t = \left[ \frac{\mu_t - r S_t}{\sigma_t} \right] dt$$

$$(15.98) \quad d[e^{-rt} S_t] = e^{-rt} \sigma_t dW_t^*$$

$$(15.111) \quad d[e^{-rt} F(S_t, t)] = e^{-rt} \sigma_t F_s dW_t^*$$

## Chapter 17

$$(page 282) \quad R_{t_1} = (1 + r_{t_1} \Delta)$$

$$(page 282) \quad R_{t_2} = (1 + r_{t_2} \Delta)$$

$$(page 282) \quad B_{t_1}^s = B(t_1, t_3)$$

$$(page 282) \quad B_{t_1} = B(t_1, T)$$

$$(page 282) \quad B_{t_3}^s = B(t_3, T)$$

$$(17.6) \quad \begin{bmatrix} 1 \\ 0 \\ B_{t_1}^s \\ B_{t_1} \\ C_{t_1} \end{bmatrix} = \begin{bmatrix} R_{t_1} R_{t_2}^u & R_{t_1} R_{t_2}^u & R_{t_1} R_{t_2}^d & R_{t_1} R_{t_2}^d \\ (F_{t_1} - L_{t_2}^u) & (F_{t_1} - L_{t_2}^u) & (F_{t_1} - L_{t_2}^d) & (F_{t_1} - L_{t_2}^d) \\ 1 & 1 & 1 & 1 \\ B_{t_3}^{uu} & B_{t_3}^{ud} & B_{t_3}^{du} & B_{t_3}^{dd} \\ C_{t_3}^{uu} & C_{t_3}^{ud} & C_{t_3}^{du} & C_{t_3}^{dd} \end{bmatrix} \begin{bmatrix} \psi^{uu} \\ \psi^{ud} \\ \psi^{du} \\ \psi^{dd} \end{bmatrix}$$

$$(17.13) \quad 1 = R_{t_1} R_{t_2}^u \psi^{uu} + R_{t_1} R_{t_2}^u \psi^{ud} + R_{t_1} R_{t_2}^d \psi^{du} + R_{t_1} R_{t_2}^d \psi^{dd}$$

$$(17.14) \quad \mathbb{Q}_{ij} = (1 + r_{t_1})(1 + r_{t_2}^i) \psi^{ij}$$

$$(17.15) \quad 1 = \mathbb{Q}_{uu} + \mathbb{Q}_{ud} + \mathbb{Q}_{du} + \mathbb{Q}_{dd}$$

$$(17.16) \quad \mathbb{Q}_{ij} > 0$$

$$(17.18) \quad B_{t_1}^s = \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{(1 + r_{t_1})(1 + r_{t_2})} \right]$$

$$(17.21) \quad B_{t_1} = \mathbb{E}^{\mathbb{Q}} \left[ \frac{B_{t_3}}{(1 + r_{t_1})(1 + r_{t_2})} \right]$$

$$(17.22) \quad 0 = \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{(1 + r_{t_1})(1 + r_{t_2})} [F_{t_1} - L_{t_2}] \right]$$

$$(17.23) \quad C_{t_1} = \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{(1 + r_{t_1})(1 + r_{t_2})} C_{t_3} \right]$$

$$(17.31) \quad F_{t_1} = \frac{1}{\mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{(1 + r_{t_1})(1 + r_{t_2})} \right]} \mathbb{E}^{\mathbb{Q}} \left[ \frac{L_{t_2}}{(1 + r_{t_1})(1 + r_{t_2})} \right]$$

$$(17.36) \quad B_{t_1}^s = \psi^{uu} + \psi^{ud} + \psi^{du} + \psi^{dd}$$

$$(17.38) \quad \pi_{ij} = \frac{1}{B_{t_1}^s} \psi^{ij}$$

$$(17.39) \quad 1 = \pi_{uu} + \pi_{ud} + \pi_{du} + \pi_{dd}$$

$$(17.46) \quad F_{t_1} = \mathbb{E}^{\pi} [L_{t_2}]$$

$$(17.52) \quad C_{t_3} = N \max[L_{t_2} - K, 0]$$

$$(17.53) \quad C_{t_1} = \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{(1 + r_{t_1})(1 + r_{t_2})} \max[L_{t_2} - K, 0] \right]$$

$$(17.55) \quad C_{t_1} = B_{t_1}^s \mathbb{E}^{\pi} [\max[L_{t_2} - K, 0]]$$

$$(17.56) \quad \frac{C_t}{S_t} = \mathbb{E}_t^{\mathbb{S}} \left( \frac{C_T}{S_T} \right)$$

$$(17.57) \quad \frac{C(K)}{S_0} = \mathbb{E}_t^{\mathbb{S}} \left( \frac{(S_T - K)^+}{S_T} \right)$$

$$\text{(page 290)} \quad y = \log \left( \frac{S_T}{K} \right)$$

$$\text{(page 291)} \quad \frac{C(K)}{S_0} = \int_0^{\infty} (1 - F(y)) e^{-y} dy$$

$$(17.63) \quad F(t; T, S) = \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right)$$

$$(17.64) \quad 0 = T_0 < T_1 < T_2 < \dots < T_M$$

$$(17.65) \quad \Delta_i = T_{i+1} - T_i, i = 0, 1, 2, \dots, M - 1$$

$$(17.66) \quad L_n(t) = \frac{P(t, T_n) - P(t, T_{n+1})}{\Delta_n P(t, T_{n+1})}$$

$$(17.71) \quad P(T_i, T_{n+1}) = \prod_{j=1}^n \frac{1}{1 + \Delta_j L_j(T_i)}$$

$$(17.72) \quad P(t, T_n) = P(t, T_l) \prod_{j=l}^{n-1} \frac{1}{1 + \Delta_j L_j(T_j)}, \quad T_{l-1} < t \leq T_l$$

$$(17.75) \quad B_t^* = P(t, T_l) \prod_{j=0}^{l-1} (1 + \Delta_j L_j(T_j))$$

$$(17.77) \quad D_n(t) = \frac{\prod_{j=l}^{n-1} \frac{1}{1 + \delta_j L_j(t)}}{\prod_{j=0}^{l-1} (1 + \Delta_j L_j(T_j))}$$

$$(17.78) \quad dL_n(t) = \mu_n(t) L_n(t) dt + L_n(t) \sigma_n^\tau(t) dW_t, \quad 0 \leq t \leq T_n, \quad n = 1, 2, \dots, M$$

$$(17.103) \quad \mu_n(t) = \sum_{j=l}^n \frac{\Delta_j L_j(t) \sigma_n^\tau(t) \sigma_j(t)}{1 + \Delta_j L_j(t)}$$

$$(17.105) \quad V_t = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^{T+\delta} r_u du} (F_t - L_T) N \delta \right]$$

$$(17.110) \quad V_t = \mathbb{E}_t^\pi [B(t, T + \delta) (F_t - L_T) N \delta]$$

$$(17.111) \quad V_t = B(t, T + \delta) \mathbb{E}_t^\pi [(F_t - L_T) N \delta]$$

$$(17.112) \quad F_t = \mathbb{E}_t^\pi [L_T]$$

$$(page 296) \quad C_T = \max[L_{T-\delta} - K, 0]$$

$$(17.113) \quad C_t = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^{T+\delta} r_u du} [L_{T-\delta} - K, 0] \right]$$

## Chapter 18

$$(18.3) \quad B(t, T) = e^{-R(t, t)(T-t)}, \quad t < T$$

$$(18.12) \quad B(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T r_s ds} \right]$$

$$(18.20) \quad R(t, T) = \frac{-\log \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T r_s ds} \right]}{T - t}$$

$$(18.33) \quad B(t, T) = e^{-\int_t^T F(t, s) ds}$$

$$(18.40) \quad F(t, T) = \lim_{\Delta \rightarrow 0} \frac{\log B(t, T) - \log B(t, T + \Delta)}{\Delta}$$

$$(page 311) \quad F(t, T, U) = \frac{\log B(t, T) - \log B(t, U)}{U - T}$$

## Chapter 19

$$(19.14) \quad dB_t = \mu(t, T, B_t) B_t dt + \sigma(t, T, B_t) B_t dV_t^T$$

$$(19.15) \quad dB_t = r_t B_t dt + \sigma(t, T, B_t) B_t dW_t^T$$

$$(19.21) \quad dF(t, T) = \sigma(t, T, B(t, T)) \left[ \frac{\partial \sigma(t, T, B(t, T))}{\partial T} \right] dt - \left[ \frac{\partial \sigma(t, T, B(t, T))}{\partial T} \right] dW_t$$

$$(19.22) \quad dF(t, T) = a(F(t, T), t) dt + b(F(t, T), t) dW_t$$

$$(19.25) \quad r_t = F(t, t)$$

(page 325) 
$$F(t, T) = F(0, T) + \int_0^t b(s, T) \left[ \int_s^T b(s, u) du \right] ds + \int_0^t b(s, T) dW_s$$

(19.26) 
$$r_t = F(0, t) + \int_0^t b(s, t) \left[ \int_s^t b(s, u) du \right] ds + \int_0^t b(s, t) dW_s$$

(19.33) 
$$dF(t, T) = b^2(T - t)dt + b dW_t$$

(19.34) 
$$dB(t, T) = r_t B(t, T)dt - b(T - t)B(t, T)dW_t$$

(19.35) 
$$r_t = F(0, t) + \frac{1}{2}b^2 t^2 + bW_t$$

(19.36) 
$$dr_t = (F_t(0, t) + b^2 t)dt + b dW_t$$

(19.37) 
$$F_t(0, t) = \frac{\partial F(0, t)}{\partial t}$$

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Chapter 6**

(6.6) 
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

(6.7) 
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0$$

(6.8) 
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f)S \frac{\partial V}{\partial S} - rV = 0$$

(6.9) 
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + q)S \frac{\partial V}{\partial S} - rV = 0$$

(6.10) 
$$\frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 \mathcal{V}}{\partial F^2} + -r\mathcal{V} = 0$$

**Chapter 8**

(8.7) 
$$V(S, t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty e^{-(\log(S/S') + (r - \frac{1}{2}\sigma^2)(T-t))^2 / 2\sigma^2(T-t)} \text{Payoff}(S') \frac{dS'}{S'}$$

<b>Call option value</b>	
	$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$
$d_1$	$= \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$
$d_2$	$= \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$
	$= d_1 - \sigma\sqrt{T-t}$

<b>Put option value</b>	
$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$	

<b>Binary call option value</b>	
$e^{-r(T-t)}N(d_2)$	

<b>Binary put option value</b>	
$e^{-r(T-t)}(1 - N(d_2))$	

**Delta of common contracts**

Call	$e^{-D(T-t)}N(d_1)$
Put	$e^{-D(T-t)}(N(d_1) - 1)$
Binary call	$\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$
Binary put	$-\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$
	$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$

**Gamma of common contracts**

Call	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Put	$-\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Binary call	$-\frac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2 S^2(T-t)}$
Binary Put	$\frac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2 S^2(T-t)}$

**Thetas of common contracts**

Call	$-\frac{\sigma S e^{-D(T-t)}N'(d_1)}{2\sqrt{T-t}} + DSN(d_1)e^{-D(T-t)} - rEe^{-r(T-t)}N(d_2)$
Put	$-\frac{\sigma S e^{-D(T-t)}N'(-d_1)}{2\sqrt{T-t}} - DSN(-d_1)e^{-D(T-t)} + rEe^{-r(T-t)}N(-d_2)$
Binary call	$re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$
Binary put	$re^{-r(T-t)}(1 - N(d_2)) - e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$

**Speed of common contracts**

Call	$-\frac{e^{-D(T-t)}N'(d_1)}{\sigma^2 S^2(T-t)}(d_1 + \sigma\sqrt{T-t})$
Put	$-\frac{e^{-D(T-t)}N'(d_1)}{\sigma^2 S^2(T-t)}(d_1 + \sigma\sqrt{T-t})$
Binary call	$-\frac{e^{-r(T-t)}N'(d_2)}{\sigma^2 S^3(T-t)}\left(-2d_1 + \frac{1-d_1d_2}{\sigma\sqrt{T-t}}\right)$
Binary put	$\frac{e^{-r(T-t)}N'(d_2)}{\sigma^2 S^3(T-t)}\left(-2d_1 + \frac{1-d_1d_2}{\sigma\sqrt{T-t}}\right)$

**Vegas of common contracts**

Call	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Put	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Binary call	$-e^{-r(T-t)}N'(d_2)\frac{d_1}{\sigma}$
Binary put	$e^{-r(T-t)}N'(d_2)\frac{d_1}{\sigma}$

**Rhos of common contracts**

Call	$E(T-t)e^{-r(T-t)}N(d_2)$
Put	$-E(T-t)e^{-r(T-t)}N(-d_2)$
Binary call	$-(T-t)e^{-r(T-t)}N(d_2) + \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$-(T-t)e^{-r(T-t)}(1 - N(d_2)) - \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

**Sensitivity to dividend for common contracts**

Call	$-(T-t)Se^{-D(T-t)}N(d_1)$
Put	$(T-t)Se^{-D(T-t)}N(-d_1)$
Binary call	$-\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

## Chapter 10

Section 10.4, one time step mark-to-market profit (using Black-Scholes with  $\sigma = \tilde{\sigma}$ )

$$= \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt + (\Delta^i - \Delta^a)((\mu - r + D)Sdt + \sigma SdX)$$

Section 10.5, mark-to-market profit from today to tomorrow

$$= \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt$$

## Chapter 14

$$(14.1) \quad V = Pe^{-y(T-t)} + \sum_{i=1}^N C_i e^{-y(t_i-t)}$$

## Chapter 16

$$(16.4) \quad \frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0$$

$$(16.5) \quad dV - rVdt = w \frac{\partial V}{\partial r} (dX + \lambda dt)$$

### 16.6 NAMED MODELS

#### 16.6.1 Vasicek

$$dr = (\eta - \gamma r)dt + \beta^{1/2}dX$$

The value of a zero coupon bond is given by  $e^{A(t;T) - rB(t;T)}$

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

$$A = \frac{1}{\gamma^2}(B(t;T) - T + t)(\eta\gamma - \frac{1}{2}\beta) - \frac{\beta B(t;T)^2}{4\gamma}$$

#### 16.6.2 Cox, Ingersoll & Ross

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX$$

The value of a zero coupon bond is given by  $e^{A(t;T) - rB(t;T)}$

$$\frac{\alpha}{2}A = a\psi_2 \log(a - B) + \psi_2 b \log((B + b)/b) - a\psi_2 \log a$$

$$B(t;T) = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1}$$

$$\psi_1 = \sqrt{\gamma^2 + 2\alpha} \text{ and } \psi_2 = \frac{\eta}{a + b}$$

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}$$

#### 16.6.3 Ho & Lee

$$dr = \eta(t)dt + \beta^{1/2}dX$$



The value of a zero coupon bond is given by  $e^{A(t;T)-rB(t;T)}$

$$B = T - t$$

$$A = - \int_t^T \eta(s)(T-s)ds + \frac{1}{6}\beta(T-t)^3$$

$$\eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + \beta(t-t^*)$$

### Section 16.7

$$\text{Forward price} = \frac{S}{Z}$$

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w) \frac{\partial Z}{\partial r} - rZ = 0$$

$$\text{with } Z(r, T) = 1$$

### Section 16.8

$$\text{Futures price } F(S, r, t) = \frac{S}{p(r, t)}$$

$$\frac{\partial p}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 p}{\partial r^2} + (\mu - \lambda w) \frac{\partial p}{\partial r} - rp - w^2 \frac{\left(\frac{\partial p}{\partial r}\right)^2}{q} + p\sigma\beta \frac{\partial p}{\partial r} = 0$$

$$\text{with } p(r, T) = 1$$

## Chapter 17

### Section 17.2 Ho & Lee

$$dr = \eta(t)dt + cdX$$

$$Z(r, t : T) = e^{A(t;T)-r(T-t)}$$

$$A(t; T) = \log \left( \frac{Z_M(t^*; T)}{Z_M(t^*; t)} \right) - (T-t) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{1}{2}c^2(t-t^*)(T-t)^2$$

### Section 17.3 The Extended Vasicek Model of Hull & White

$$dr = (\eta - \gamma r)dt + cdX$$

$$dr = (\eta(t) - \gamma r)dt + cdX$$

$$Z(r, t : T) = e^{A(t;T)-rB(t;T)}$$

$$B(t; T) = \frac{1}{\gamma} \left( 1 - e^{-\gamma(T-t)} \right)$$

$$A(t; T) = \log \left( \frac{Z_M(t^*; T)}{Z_M(t^*; t)} \right) - B(t; T) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{c^2}{4\gamma^3} \left( e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)} \right)^2 \left( e^{2\gamma(t-t^*)} - 1 \right)$$

## Chapter 18

### Section 18.3.1 Bond Options - Market Practice

Call option price:  $e^{-r(T-t)}(FN(d'_1) - EN(d'_2))$

Put option price:  $e^{-r(T-t)}(EN(-d'_2) - FN(-d'_1))$

$$d'_1 = \frac{\log(F/E) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d'_2 = d'_1 - \sigma\sqrt{T-t}$$

### Section 18.4.3 Caps and Floors - Market Practice

Caplet price:  $e^{-r^*(t_i-t)}(F(t, t_{i-1}, t_i)N(d'_1) - r_c N(d'_2))$

Floorlet price:  $e^{-r^*(t_i-t)}(-F(t, t_{i-1}, t_i)N(-d'_1) + r_c N(-d'_2))$

$$d'_1 = \frac{\log(F/r_c) + \frac{1}{2}\sigma^2 t_{i-1}}{\sigma\sqrt{t_{i-1}}}$$

$$d'_2 = d'_1 - \sigma\sqrt{t_{i-1}}$$

### Section 18.6.1 Swaptions, Captions and Floortions - Market Practice

Price of a payer swaption:  $\frac{1}{F}e^{-r(T-t)}\left(1 - \left(1 + \frac{1}{2}F\right)^{-2(T_s-T)}\right)(FN(d'_1) - EN(d'_2))$

Price of a receiver swaption:  $\frac{1}{F}e^{-r(T-t)}\left(1 - \left(1 + \frac{1}{2}F\right)^{-2(T_s-T)}\right)(EN(-d'_2) - FN(-d'_1))$

$$d'_1 = \frac{\log(F/E) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d'_2 = d'_1 - \sigma\sqrt{T-t}$$

## Chapter 19

$$(19.1) \quad Z(t; T) = e^{-\int_t^T F(t; s) ds}$$

$$(19.2) \quad dZ(t; T) = \mu(t, T)Z(t; T)dt + \sigma(t, T)Z(t; T)dX$$

$$(nn) \quad F(t; T) = -\frac{\partial}{\partial T} \log Z(t; T)$$

$$(19.3) \quad dF(t; T) = \frac{\partial}{\partial T} \left( \frac{1}{2} \sigma^2(t, T) - \mu(t, T) \right) dt - \frac{\partial}{\partial T} \sigma(t, T) dX$$

$$(19.5) \quad dF(t; T) = \nu(t, T) \left( \int_t^T \nu(t, s) ds \right) dt + \nu(t, T) dX$$

$$(19.6) \quad dF(t; T) = \left( \sum_{i=1}^N \nu_i(t, T) \int_t^T \nu_i(t, s) ds \right) dt + \sum_{i=1}^N \nu_i(t, T) dX_i$$

$$(19.7) \quad dZ_i = rZ_i dt + Z_i \sum_{j=1}^{i-1} a_{ij} dX_j$$

$$(19.8) \quad dZ_i = (1 + \tau F_i) dZ_{i+1} + \tau Z_{i+1} dF_i + \tau \sigma_i F_i Z_{i+1} \left( \sum_{j=1}^i a_{i+1, j} \rho_{ij} \right) dt$$

$$(19.9) \quad dF_i = \left( \sum_{j=1}^i \frac{\sigma_j F_j \tau \rho_{ij}}{1 + \tau F_j} \right) \sigma_i F_i dt + \sigma_i F_i dX_i$$

**The Handbook of Fixed Income Securities, 8th ed., F. Fabozzi**

Page 202  $Y_d = \frac{(F - P)}{F} \times \frac{360}{t}$

**Managing Investment Portfolios, a dynamic process, Maginn, et al  
Chapter 5**

$$(5-1) \quad U_m = E(R_m) - 0.005 R_A \sigma_m^2$$

$$(5-2) \quad \text{SFRatio} = \frac{E(R_P) - R_L}{\sigma_P}$$

$$(5-3) \quad \frac{E(R_{new}) - R_F}{\sigma_{new}} > \left( \frac{E(R_p) - R_F}{\sigma_p} \right) \text{Corr}(R_{new}, R_p)$$

$$(5-4) \quad U_m^{\text{ALM}} = E(\text{SR}_m) - 0.005 R_A \sigma^2(\text{SR}_m)$$

**Analysis of Financial Time Series, third edition, R. Tsay**  
**Chapter 3**

$$(3.2) \quad \mu_t = E(r_t|F_{t-1}), \quad \sigma_t^2 = Var(r_t|F_{t-1}) = E[(r_t - \mu_t)^2|F_{t-1}]$$

$$(3.3) \quad r_t = \mu_t + a_t, \quad \mu_t = \sum_{i=1}^p \phi_i y_{t-i} - \sum_{i=1}^q \theta_i a_{t-i}, \quad y_t = r_t - \phi_0 - \sum_{i=1}^k \beta_i x_{it}$$

$$(3.4) \quad \sigma_t^2 = Var(r_t|F_{t-1}) = Var(a_t|F_{t-1})$$

$$(p114) \quad a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + e_t. \quad t = m+1, \dots, T$$

$$(p114) \quad F = \frac{(SSR_0 - SSR_1)/m}{SSR_1/(T - 2m - 1)}$$

$$(3.5) \quad a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2$$

$$(p117) \quad a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2$$

$$(p118) \quad E(a_t^4) = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)}$$

$$(p120) \quad f(a_1, \dots, a_T|\alpha) = f(a_T|F_{T-1})f(a_{T-1}|F_{T-2}) \cdots f(a_{m+1}|F_m)f(a_1, \dots, a_m|\alpha) \\ = \prod_{t=m+1}^T \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{a_t^2}{2\sigma_t^2}\right) \times f(a_1, \dots, a_m|\alpha)$$

$$(p121) \quad \ell(a_{m+1}, \dots, a_T|\alpha, \alpha_1, \dots, \alpha_m) = - \sum_{t=m+1}^T \left[ \frac{1}{2} \ln(\sigma_t^2) + \frac{1}{2} \frac{a_t^2}{\sigma_t^2} \right]$$

$$(3.7) \quad f(\epsilon_t|v) = \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(1 + \frac{\epsilon_t^2}{v-2}\right)^{-(v+1)/2}, \quad v > 2$$

$$(3.8) \quad \ell(a_{m+1}, \dots, a_T|\alpha, A_m) = - \sum_{t=m+1}^T \left[ \frac{v+1}{2} \ln\left(1 + \frac{a_t^2}{(v-2)\sigma_t^2}\right) + \frac{1}{2} \ln(\sigma_t^2) \right]$$

$$(p121) \quad \ell(a_{m+1}, \dots, a_T|\alpha, v, A_m) = (T - m) \left\{ \ln \left[ \Gamma\left(\frac{v+1}{2}\right) \right] - \ln \left[ \Gamma\left(\frac{v}{2}\right) \right] \right. \\ \left. - 0.5 \ln[(v-2)\pi] \right\} + \ell(a_{m+1}, \dots, a_T|\alpha, A_m)$$

$$(3.9) \quad g(\epsilon_t|\xi, v) = \begin{cases} \frac{2}{\xi + \frac{1}{\xi}} \varrho f[\xi(\varrho\epsilon_t + \bar{\omega})|v] & \text{if } \epsilon_t < -\bar{\omega}/\varrho \\ \frac{2}{\xi + \frac{1}{\xi}} \varrho f[(\varrho\epsilon_t + \bar{\omega})/\xi|v] & \text{if } \epsilon_t \geq -\bar{\omega}/\varrho \end{cases}$$

(p122) 
$$\varpi = \frac{\Gamma[(v-1)/2]\sqrt{v-2}}{\sqrt{\pi}\Gamma(v/2)} \left( \xi - \frac{1}{\xi} \right), \quad \varrho^2 = \left( \xi^2 + \frac{1}{\xi^2} - 1 \right) - \varpi^2$$

(3.10) 
$$f(x) = \frac{v \exp(-\frac{1}{2}|x/\lambda|^v)}{\lambda 2^{(1+1/v)}\Gamma(1/v)}, \quad -\infty < x < \infty, \quad 0 < v \leq \infty$$

(3.11) 
$$\sigma_h^2(\ell) = \alpha_0 + \sum_{i=1}^m \alpha_i \sigma_h^2(\ell - i). \quad \text{where } \sigma_h^2(\ell - i) = a_{h+\ell-i}^2 \text{ if } \ell - i \leq 0$$

(3.14) 
$$GARCH(m, s) : a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2$$

(3.15) 
$$GARCH(m, s) : a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^s \beta_j \eta_{t-j}$$

(p132) 
$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3$$

(3.17) 
$$\sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(\ell - 1), \quad \ell > 1$$

(p133) 
$$\sigma_h^2(\ell) = \frac{\alpha_0 [1 - (\alpha_1 + \beta_1)^{\ell-1}]}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)^{\ell-1} \sigma_h^2(1)$$

(3.22) 
$$\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1$$

(3.23) 
$$GARCH(1, 1) - M : r_t = \mu + c\sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

(3.24) 
$$g(\epsilon_t) = \theta \epsilon_t + \gamma [|\epsilon_t| - E(|\epsilon_t|)]$$

(p143) 
$$g(\epsilon_t) = \begin{cases} (\theta + \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0 \\ (\theta - \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0 \end{cases}$$

(p143) 
$$E(|\epsilon_t|) = \frac{2\sqrt{v-2}\Gamma[(v+1)/2]}{(v-1)\Gamma(v/2)\sqrt{\pi}}$$

(3.25) 
$$EGARCH(m, s) : a_t = \sigma_t \epsilon_t, \quad \ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \dots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \dots - \alpha_m B^m} g(\epsilon_{t-1})$$

(3.26) 
$$a_t = \sigma_t \epsilon_t, \quad (1 - \alpha B) \ln(\sigma_t^2) = (1 - \alpha)\alpha_0 + g(\epsilon_{t-1})$$

(3.27) 
$$(1 - \alpha B) \ln(\sigma_t^2) = \begin{cases} \alpha_* + (\gamma + \theta)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0 \\ \alpha_* + (\gamma - \theta)(-\epsilon_{t-1}) & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

(p144) 
$$\sigma_t^2 = \sigma_{t-1}^{2\alpha_*} \exp(\alpha_*) \begin{cases} \exp\left[ (\gamma + \theta) \frac{a_{t-1}}{\sigma_{t-1}} \right] & \text{if } a_{t-1} \geq 0 \\ \exp\left[ (\gamma - \theta) \frac{|a_{t-1}|}{\sigma_{t-1}} \right] & \text{if } a_{t-1} < 0 \end{cases}$$

(p148) 
$$\sigma_{h+1}^2 = \sigma_h^{2\alpha_1} \exp[(1 - \alpha_1)\alpha_0] \exp[g(\epsilon_{h+1})]$$

(p148) 
$$E\{\exp[g(\epsilon)]\} = \exp\left(-\gamma\sqrt{2/\pi}\right) \left[ e^{(\theta+\gamma)^2/2} \Phi(\theta + \gamma) + e^{(\theta-\gamma)^2/2} \Phi(\gamma - \theta) \right]$$

(p148) 
$$\hat{\sigma}_h^2(j) = \hat{\sigma}_h^{2\alpha_1}(j-1) \exp(\omega) \left\{ \exp[(\theta + \gamma)^2/2] \Phi(\theta + \gamma) + \exp[(\theta - \gamma)^2/2] \Phi(\gamma - \theta) \right\}$$

**Frequently Asked Questions in Quantative Finance, P. Wilmott (QFIC-113-17)**

**Q23 - Jensen's Inequality (103-105)**

If the payoff is a convex function,  $E[P(S_T)] \geq P(E[S_T])$

$$\begin{aligned} E[f(S)] &= E[f(\bar{S} + \epsilon)] = E\left[f(\bar{S}) + \epsilon f'(\bar{S}) + \frac{1}{2}\epsilon^2 f''(\bar{S}) + \dots\right] \\ &\approx f(\bar{S}) + \frac{1}{2}f''(\bar{S})E[\epsilon^2] \\ &= f(E[S]) + \frac{1}{2}f''(E[S])E[\epsilon^2] \end{aligned}$$

The LHS is greater than the RHS by approximately  $\frac{1}{2}f''(E[S])E[\epsilon^2]$

**Q26 - Girsanov's Theorem (113-115)** The Theorem is:

Let  $W_t$  be a Brownian motion with measure  $\mathbb{P}$  and sample space  $\Omega$ . If  $\gamma_t$  is a previsible process satisfying the constraint  $E_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2\right)\right] < \infty$  then there exists an equivalent measure  $\mathbb{Q}$  on  $\Omega$  such that  $\tilde{W}_t = W_t + \int_0^t \gamma_s ds$  is a Brownian motion.

Chapter 9-Volatility

$$(9.1) \quad \text{Prob}(v > x) = Kx^{-\alpha}$$

$$(9.5) \quad \sigma_n^2 = \sum_{i=1}^m \alpha_i u_{n-i}^2$$

(page 186) 
$$\sum_{i=1}^m \alpha_i = 1$$

$$(9.6) \quad \sigma_n^2 = \gamma V_L + \sum_{i=1}^m \alpha_i u_{n-i}^2$$

$$(9.7) \quad \sigma_n^2 = \omega + \sum_{i=1}^m \alpha_i u_{n-i}^2$$

$$(9.8) \quad \sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2$$

$$(9.9) \quad \sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

$$(9.10) \quad \sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

$$(9.12) \quad \sum_{i=1}^m \left[ -\ln v - \frac{u_i^2}{v} \right]$$

$$(9.13) \quad \sum_{i=1}^m \left[ -\ln(v_i) - \frac{u_i^2}{v_i} \right]$$

$$(9.14) \quad E[\sigma_{n+t}^2] = V_L + (\alpha + \beta)^t (\sigma_n^2 - V_L)$$

$$(9.15) \quad \sigma(T)^2 = 252 \left\{ V_L + \frac{1 - e^{-aT}}{aT} [V(0) - V_L] \right\}$$

$$(9.16) \quad \frac{\Delta \sigma(T)}{\Delta \sigma(0)} \approx \frac{1 - e^{-aT}}{aT} \frac{\sigma(0)}{\sigma(T)}$$