Copula Phase Transitions

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Abstract. This work represents two related but distinctly different purposes. The first is to incorporate phase transitions into a copula framework. Phase transitions are a concept common to statistical physics and probability theory in which a phenomenological behavior changes abruptly and often, but not always, as the result of a parametric change. The second purpose is to calculate the probability of defaulting in the context of a portfolio of correlated mortgages when a contagion effect exists. Understanding and measuring the effects that contagion has on the probability of defaulting, for both single assets and portfolios of assets, are crucial to CDO markets.

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1 Introduction

Although the mortgage crisis of 2007-08 was well in progress before this work begun, and these events are nearly perfect for illustrating the principles that underlie phase transitions, the methods are not specific to these events. Phase transitions are useful for analyzing the problem of correlated defaults that are undergoing contagion.

There are several ways that the concept of phase transitions may be introduced, and this paper aims to introduce one that we believe will be useful to practitioners. The author tries to take a very pragmatic viewpoint: assume that the mortgage crisis had not yet occurred and that one were trying to compute probabilities of mortgages defaulting. The problem of correlated defaulting is fundamental to pricing of security derivatives backed by mortgages.

A general framework for correlated defaulting for collateralized debt obligations (CDOs) was expounded upon by David Li (see [1]) in his paper On Default Correlation: A Copula Function Approach. One of the most important attributes of this copula approach to modeling correlated defaulting is that one can compute in a practical manner from the approach Li put forth. Li's original paper concerned CDOs constructed from corporate loans and not mortgages, and although there is nothing specific about the method that would limit its use to only corporate loans, one should not think that residential mortgages and corporate loans are equivalent, but they are not dissimilar either. Whether one is concerned with corporate loans or residential mortgages, the use of optimistic inputs into the method will yield optimistic predictions.

To be specific, Li's method requires two inputs

1. The probability of default for individual assets.
2. The correlation between assets.

Correlation between assets is a complicated matter, because the correlations between assets are not static.

The phase transition behavior that will be introduced is basically the infectious/contagion spread throughout markets. If the contagion is small and limited in scope, then the asset pool is in a particularly good phase. If the contagion is relatively large, then the phase is considered to be in the bad phase. For this setting, we consider only two phases: good and bad. It may prove that bad and horrific would be more appropriate names and also, more than two phases might be appropriate.

Rather than producing an a priori estimate of such phase transitions, we consider the problem of determining whether a phase transition (or contagion) is occurring and how much lead time one might have before the contagion's effect becomes wide-spread.

Consider the following scenario concerning a market for which a bubble has formed. As the bubble is forming there is opportunity for profit, but all bubbles break and while it is expanding it is quite likely that one does not have all the data required to correctly predict when the bubble will reach a breaking point or when it will stop expanding, reaching an inflection point. Risk managers may need to proceed even in the advent of risk that may in all actuality still be ill-posed and extreme. Ill-posed does not refer to the inherent randomness associated with risk, but rather that models are never complete and even when they are sufficient (and hopefully useful), the model inputs need to be correctly specified. If one has taken a completely indefensible position and the required inputs to a model are lacking, then although a well-posed model may be completely defendable, the real life problem may still be ill-posed. One may argue that all real world problems are never and will never be truly well-posed and in this regard, the real world works on plausible arguments and not on proofs. The use of overly optimistic inputs will never support plausibility.

Furthermore and on a different chain of thought, an a priori model for a copula phase transition could be well-posed, but may be of limited use as a quantitative model, because, in part, its sensitivity to initial conditions may render such a model difficult to use and also lead to a false sense of security. The effort is still potentially useful because the scenarios generated may be instructive.

For this reason, our work has focused on quantitative estimates to measure the degree to which a priori assumptions/inputs are not being realized and the degree to which a portfolio of assets is defaulting faster than expected, whether due to contagion or overly optimistic choices for correlation or individual asset default probabilities.

2 Key inputs to the copula approach

We begin by summarizing David Li's copula approach to correlated defaulting. There is nothing particular that requires that the copula be a Gaussian copula, but for concreteness we will use the Gaussian copula.

2.1 Copulas

The use of copulas in statistics is an ansatz method that postulates the functional form that a multivariate distribution shall have. The copula is specifically a multi-dimensional function $C : [0,1]^n \rightarrow [0,1]$, with the following properties:

- $C(\boldsymbol{u}) = 0$, if $\boldsymbol{u} \in [0,1]^n$ has at least one component equal to 0.
- $C(\boldsymbol{u}) = u_i$, if $\boldsymbol{u} \in [0,1]^n$ has every component equal to 1 except the i^{th} component that equals u_i.

Suppose that τ_A and τ_B are random variables for the default times of assets A and B, respectively with the probability distributions $P(\tau_A < t)$ and $P(\tau_B < t)$. The copula (Gaussian) ansatz states that the joint distribution for both defaulting is given by

$$P(\tau_A < t, \tau_B < t) = \Phi_2(\Phi^{-1}(P(\tau_A < t)), \Phi^{-1}(P(\tau_B < t)), \rho_{AB}).$$

The function Φ_2 is the bivariate normal distribution with mean 0 and covariance ρ_{AB}. The function Φ^{-1} is the inverse of the normal distribution with mean 0 and variance 1.

There are two main components to Li's approach to correlated defaulting. In the context of mortgages, the probability that a single mortgage/asset defaults is needed and additionally, the correlation between mortgages is needed. One can easily imagine the probability of a mortgage defaulting depending on several parameters. Originally, this work had hoped for data that might be data mined to evolve such a notion of dependency for default time as well as correlation between assets, but the reality of acquiring data has not been simple. As such, a more minimalist point of view was taken: What is the least amount of data needed to be able to compute and study correlated defaulting in the context of mortgages and develop an approach to copula phase transitions. Even in this case there are choices. For example, does one use FICO scores or loan-to-value ratios? In the next two sub-sections, the copula approach of Li is outlined with choices that are subjective, but hopefully rational/plausible and are required before any consideration of phase transitions due to contagion can be made.

2.2 Hazard Functions

The hazard function, $h(\cdot)$, is the infinitesimal change in probability and the probability of defaulting before time t is given by

$$P(\tau < t) = 1 - e^{-\int_0^t h(s)ds}.$$

We wish to describe the probability of defaulting as a function of loan-to-value. This choice is largely based on the availability of historical data that can be used to calibrate the model. This is particularly important, because the sub-prime mortgage crisis has associated loans with extremely high loan-to-value ratios and a functional form with dependency on loan-to-value ratios is needed. We begin by making the following 2 assumptions.

1. If the loan is 0, then there is no chance for the loan to default at any time and the loan-to-value is 0, but more important $P(\tau > t) = 1$, for all $t > 0$.
2. If the loan is for the full value of the loan or loan-to-value, l, equals 1, then there was no money down on the loan. In cases that have little money down, the loan is obviously considered risky, and as $l \to 1$, the loan becomes increasingly more risky. As such, consider the idealization that as $l \to 1$ the loan instantly defaults. Although this is clearly not true, its not an unreasonable assumption for several reasons, but the most compelling argument is how well this assumption fits to data for values of $l \neq 1$. The probability of instantly defaulting can be expressed as $P(\tau > t) = 0$, for all $t > 0$. The case of loan-to-value $= 1$ will be a singularity and in practice one will need to convolve the distribution with a smoothing function or at minimum truncate the singularity. In addition, we are not addressing the case that loan-to-value exceeds 1, but it will be clear that the approach can be extended, but the question as to “how risky” are such loans needs to be addressed.

The function $\frac{l}{1-l}$ will be used to interpolate between cases 1 and 2, described above, giving a parameterized family of exponential distributions for the probability of defaulting on a loan as a function of time and initial loan-to-value. The function $\frac{l}{1-l}$ is not the only function that one could have used, but it is the simplest ratio of two linear functions having the desired boundary behavior. Let loan-to-value be denoted by $l2v$ and β be a constant that will be fit to data. The tail distribution for defaulting will have the form

$$P(\tau > t|l2v = l) = e^{-\beta\cdot(\frac{l}{1-l})\cdot t}.$$

Case 1: $l = 0$ and

$$P(\tau > t|l2v = 0) = e^{-\beta\cdot 0\cdot t} = 1, \quad \forall t > 0.$$

Case 1: $l = 1$ and

$$P(\tau > t|l2v = 1) = e^{-\beta\cdot\infty\cdot t} = 0, \quad \forall t > 0.$$

The parameter β needs to be fit to historical data. Although historical data can be erroneous, because it only represents empirical samples of a distribution, one should not undervalue that historical data represents the samples that have actually occurred.

Discussions with Dr. Thomas Herzog, Chief Actuary for the U.S. Department of Housing and Urban Development (HUD) have been illuminating in several respects, but most important, Actuaries at HUD published results that are needed for fitting β. Table 1 is taken directly from “15 Million Mortgages, The FHA Experience” (see [2]). Consider minimizing with respect to β, where the summation is over the loan-to-value categories. Taking the derivative with respect to β and setting equal to 0 gives

$$\beta\{(49)^2 + (24)^2 + (12.33)^2 + (5.66)^2\} =$$
$$\{49 * 0.1931 + 24 * 0.1159 + 12.33 * 0.0652 + 5.66 * 0.0751\}$$

$$\beta = \frac{13.46}{3161.07} = 0.004$$

Table 1. claim rates for mortgages endorsed in 1981

Claim rates for mortgages endorsed in 1981						
Loan2value %	15k-25k	25k-35k	35k-50k	50k-60k	Over	Total
Under 80%	3.07%	2.30%	2.29%	3.90%	3.47%	2.89%
80.0 89.9	10.49	8.53	7.03	8.2	5.4	7.51
90.0 94.9	11.42	9.16	6.21	5.22	4.59	6.52
95.0 96.9	13.87	14.85	11.34	10.45	8.74	11.59
97.0 98.9	17.57	25.33	19.40	14.71	6.96	19.31
Totals	10.20	10.00	7.46	7.24	5.74	7.79

The tail probability for defaulting on a loan conditioned on loan-to-value ratio needed as input to the copula method will be taken as

$$P(\tau > t | l2v = l) = e^{-0.004 \cdot (\frac{l}{1-l}) \cdot t}.$$

For the reader's convenience, Table 2 tabulates the rates against Table 1.

Table 2. Comparison of defaulting rates

Comparison of defaulting rates					
Loan2value l	0.98	0.96	0.925	0.85	0.80
$\eta(l) = (\frac{l}{1-l})$	49	24	12.33	5.66	4
$0.004 * \eta(l)$	0.196	0.096	0.049	0.023	0.016
Totals 1981	0.1931	0.1159	0.0652	0.0751	0.0289

Some observations are in order. First, there is a definite bump in the historical distribution in the loan-to-value range of 80.0-89.9. Dr. Herzog has conveyed that this bump was attributable to a series of defaulted loans that were not owner-occupied. Other than this bump, the postulated distribution compares well against the empirical default rates, but for the most part, underestimates the rate of defaulting. Although we are pleased with this functional form as a starting point, we believe that further refinement would be useful. For example, refinement by conditioning on location and other relevant attributes may result in a more useful prediction to practitioners in the credit markets.

2.3 Correlation Functions

The copula approach requires knowledge of the correlation between assets, or mortgages in this case, be known. This is not as simple a matter as one might wish to think. In the best of situations, there are different correlation functions that may be used, but in an attempt to be as realistic as possible: it is not clear in the absence of data that one can even justify any choice of correlation function. Regardless, a correlation function is needed.

Consider a pair of mortgages, and that we only know the initial loan-to-value ratios (l_A, l_B). Initially, consider the assumption that if $l_A \to l_B$, then the correlation $\rho_{AB} \to 1$ and if $|l_A - l_B| \to 1$, then ρ_{AB} should decrease, but not necessarily tend to 0. If one has no additional information, then there are several choices for functions that will have the desired boundary behavior, but some are more optimistic than others. We give several examples.

1. $\cos(\frac{\pi}{2}(l_A - l_B))$
2. $1 - |l_A - l_B|^2$
3. $1 - |l_A - l_B|$
4. $1 - \sqrt{|l_A - l_B|}$
5. $\frac{1}{4}\{e^{-3(l_A - l_B)^2} + 3e^{-4(l_A - l_B)^4}\}$

The rationale for the 1^{st} is that a Pearson correlation has a geometric interpretation. The 2^{nd} is a simpler function, but still related to the functional behavior of the 1^{st}. The 3^{rd} and 4^{th} are both optimistic choices for functional dependencies, with the 4^{th} being more optimistic than the 3^{rd} function. The 5^{th} function is interesting because of the cosine-like behavior near a loan-to-value difference of 0, but does not go to zero near a difference of 1. As such, the correlation between two assets stays strictly positive.

Note that if a pool of mortgages is mostly formed with high loan-to-value ratios, then the defaulting is twice jeopardized. First, because the individual assets have a very high probability of defaulting, and the second, a correlation function other than the 3^{rd} and 4^{th} are not overly optimistic.

3 Contagion effect: results in a chain-reaction

As defaults occur, the degree to which contagion occurs needs to be measured. Consider the additional factors that one may have been overly optimistic and under-estimated the correlation function, or the fundamentals of an individual loan defaulting may have changed in the last 25 years since the data used to fit the hazard function. In an attempt to be as practical as possible, one may ask whether it actually matters. If a portfolio of assets is defaulting quicker than expected, all that matters is determining the increased rate and the function that best describes the change in rate.

Nothing travels faster than bad news and for this reason we consider contagion to be like an infectious disease. Note that every time a mortgage defaults the associated property is often available at a distressed price which advocates a buyers market that further effects market prices. A nearby property may have an increased likelihood of defaulting due to proximity. In one respect, the effective loan-to-value ratio is increased. Secondly, nearby sellers may have been expecting or have needed to sell at minimum prices in order to avoid defaulting.

Regardless, defaulting spreads via contact, like influenza through a population. Although contact now may be considered in a broader sense of the word, we hope to aggregate all the complex behavior into a computable parameter. This parameter will encode the notion of a phase transition. If the parameter remains small and negligible then one is in the good or at least the expected phase, and if the parameter is not negligible then the behavior will enter the bad or unexpected phase.

Stochastic models associated with infectious diseases make crucial use of branching processes. Branching processes are one of the basic building blocks for studying stochastic processes that exhibit phase transitions. We begin by providing a brief primer on branching processes.

3.1 Branching Processes (Galton-Watson)

Although we will be interested in time-dependent branching processes, we start with the discrete time Galton-Watson process. Let $\xi_j^k, j, k \geq 0$ be a collection of independent identically distributed non-negative integer-valued random variables. The off-spring is given by the family of probability distributions $P(\xi_j^k = l)$. Given the off-spring distribution ξ_j^n, one iteratively constructs the size of a population at discrete time $n+1$ using the off-spring of the population at time n. The Galton-Walton process is defined by the following formula

$$Z_{n+1} = \begin{cases} \sum_{j=1}^{Z_n} \xi_j^{n+1} & : \quad if \quad Z_n > 0 \\ 0 & : \quad if \quad Z_n = 0 \end{cases}$$

A realization of this branching process is seen in Figure 1. The population associated with this realization grows as follows: $Z_1 = 1, Z_2 = 3, Z_3 = 7$, with $\xi_1^2 = 3, \xi_2^3 = 2$ and $\xi_3^3 = 3$.

Useful facts concerning the Galton-Watson process are:

- (Sub-critical regime) if $\mu = E(\xi_j^n) < 1$ then $Z_n \to 0$ exponentially fast.
- (Super-critical regime) If $\mu > 0$ then $P(Z_n > 0, \text{for all } n) > 0$.

There is also the case when $\mu = 1$ (critical regime), but we will not deal with this case. This process is the starting point for understanding many processes which have phase transitions and also

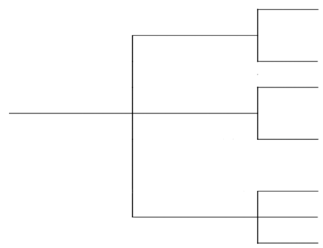

Fig. 1. depiction of branching process

understanding the stochastic spread of infections. **A key idea is that if one can identify a behavior that can be thought of as a branching behavior, then there is possibility of a phase transition**. If the branching rate is sufficiently large then the behavior is super-critical and if the branching rate is sufficiently small then the behavior is sub-critical.

3.2 Malthusian parameter

The discrete time branching has a direct extension to continuous time branching processes in which the holding time between branching events has a general holding distribution and is characterized by the Bellman-Harris process. The Malthusian parameter associated with the Bellman-Harris process is a characterization of the exponential growth or decay of the process, whereby establishing the super and sub critical regimes.

Following Theodore Harris' book The Theory of Branching Processes (see [3]), Chapter VI on age-dependent branching processes, let the Laplace transform of the holding time be given by

$$\mathcal{L}_\alpha(G) = \int_0^\infty e^{-\alpha y} dG(y).$$

Theorem 17.1 (T. Harris, chapter 6)

Suppose $m = \frac{d}{d\theta}|_{\theta=1} f(\theta) > 1$. *Define the constant* α *as the positive root of the equation*

$$m\mathcal{L}_\alpha(G) = 1 \qquad (17.1)$$

If G is not a lattice distribution then

$$E(Z_t) \approx n_1 e^{\alpha t}, \quad t \to \infty,$$

Where

$$n_1 = \frac{m-1}{-\alpha m^2 \frac{d}{d\alpha}\mathcal{L}_\alpha(G)}$$

Theorem 17.3 (T. Harris)

Suppose $m < 1$, G *is not a lattice distribution, and there exists a real* α *(necessarily negative) satisfying (17.1). Suppose in addition that* $-\frac{d}{d\alpha}\mathcal{L}_\alpha(G) < \infty$ *Then (17.2) holds with again defined by (17.3).*

Theorem 17.1 and 17.3 characterize the super and sub critical behavior of $E(Z_t)$ entirely in terms of the mean of the off-spring distribution and the Laplace transform of the time to infect distribution.

3.3 Exponential holding time case

Let $P(T > t) = e^{-\beta t}, \beta > 0$. Then $G(t) = 1 - e^{-\beta t}$ and $dG(t) = \beta e^{-\beta t}$.

$$\mathcal{L}_\alpha(G) = \int_0^\infty e^{-\alpha y} dG(y) = \int_0^\infty \beta e^{-(\alpha+\beta)y} dy = \frac{\beta}{\alpha+\beta}$$

$$m\mathcal{L}_\alpha(G) = 1 \quad \Longrightarrow \quad m\beta = \alpha = \beta \quad \Longrightarrow \quad \frac{m-1}{\alpha} = \frac{1}{\beta}$$

$$\frac{d}{d\alpha}\mathcal{L}_\alpha(G) = \frac{\beta}{(\alpha+\beta)^2}$$

$$n_1 = \frac{m-1}{-\alpha m^2 \frac{d}{d\alpha}\mathcal{L}_\alpha(G)} = \frac{(m-1)(\alpha+\beta)^2}{\alpha m^2} = \frac{1}{\beta m^2}\frac{(m\beta)^2}{\beta} = 1$$

$$E(Z_t) \approx e^{(m-1)\beta t}, t \to \infty$$

If we start by assuming a Markovian case, with exponential holding times, then $n_1 = 1$ and $E(Z_t) \approx e^{\alpha t}$.

4 Augmenting the hazard function

Consider that the hazard rate may be perturbed by the degree of contagion. For given $\alpha > 0$, let N be the number of mortgages in a pool that is experiencing contagion. At time s, the ratio $\frac{e^{\alpha s}}{N}$ is the percentage of mortgages experiencing contagion. Although the following is not the only functional dependency that is possible, consider a simple scenario in which a small percentage of contagion causes a linear perturbation to the hazard function and increases dramatically as the percentage increases. Consider that one may postulate the hazard function perturbation to again make use of the function $\frac{x}{1-x}$. There are two reasons to consider using this function again. First, it has certain desirable properties and second leads to nice formulas. Although it is unreasonable to expect the real world to be in complete agreement with this ansatz, it seemed to be amenable concerning the hazard function.

$$P(\tau < t|l2v = l) \approx 1 - e^{-\int_0^t [0.004\eta(l)+\eta(\frac{e^{\alpha s}}{N})]ds}$$

After some arithmetic and calculus, the following correction to the single asset default probability is

$$P(\tau < t|l2v = l) \approx 1 - e^{-0.004\frac{l}{1-l}t} \times \left[\frac{N - e^{\alpha t}}{N-1}\right]^{\frac{1}{\alpha}}, \alpha > 0, e^{\alpha t} \le N$$

The contagion term has been collected into the factor

$$CONTAG(N, \alpha, t) = \left[\frac{N - e^{\alpha t}}{N - 1}\right]^{\frac{1}{\alpha}},$$

where $e^{\alpha t} \leq N$. LHopital's rule may be applied to show that when $\alpha = 0$, the contagion term equals 1 and the formula still holds for $\alpha < 0$, but $\alpha > 0$ is the case of interest.

To illustrate the effect that this contagion term has on the defaulting of a single asset, consider the graphs in Figure 3, $e^{0.1960t}$ corresponds to the 0.98 loan-to-value case, a $CONTAG(1000, 0.5, t)$ term was used.

Under this choice of contagion functionality and choice of α, the default probability is largely unaffected until after $t = 4$. After which the contagion term becomes increasingly significant and dramatically alters the tail probability.

Although α was chosen relatively small (but still positive), its effect could only be delayed and not avoided. Consider that α could be significantly larger. Now for the sake of argument, consider that $\alpha = 2.5$. As seen in Figure 4, the effect on the hazard function tail is now dramatic.
The time horizon, for $\alpha = 2.5, N = 1000$, before the contagion begins to alter the tail is nominally $t = 1$ and by $t = 1.5$ the collapse is eminent. If $N = 100$, then the time horizon is shortened to $t = 0.5$, after which the contagion effect propagates and is depicted in Figure 5.

5 Numerical Experiment

The most important cases are when $\alpha > 0$, because these are the situations in which the contagion spreads and the branching process does not die out.

The following numerical experiment will allow for the use of real data, as well as synthesized data. In the absence of real data, we begin with synthesized data, but first construct the numerical framework for the experiment.

Regardless of whether the data is real or synthesized, consider that A and B represent two assets from among N assets. Let the matrix $\delta_{AB}(t)$ denote the pair-wise indicator functions as to whether pairs of assets have both defaulted by time $t > 0$. That is

$$\delta_{AB}(t) = \begin{cases} 1 & : \ if \quad \text{both } A \text{ and } B \text{ have defaulted by time } t \\ 0 & : \ if \quad \text{otherwise} \end{cases}$$

The joint probability distribution for pair-wise defaulting is approximated by

$$\begin{aligned} p^{\alpha}_{AB} &= P(\tau_A < t, \tau_B < t | \alpha > 0) \\ &= \Phi_2(\Phi^{-1}(P(\tau_A < t|\alpha > 0)), \Phi^{-1}(P(\tau_B < t|\alpha > 0)), \rho_{AB}) \end{aligned}$$

and one can compute a discrepancy/error term for fixed t and subsequently compute the α that minimizes this discrepancy. For each $t > 0$, let $\mathcal{A}(t)$ denote the best alpha and given by the formula

$$\mathcal{A}(t) = \min_{\alpha > 0} \left\{ \sum_{AB} (\delta_{AB}(t) - p^{\alpha}_{AB})^2 \right\}$$

6 Synthesized Data

In order to synthesize data, consider fixing $\alpha > 0$ and sampling from the distribution p^{α}_{AB} for the first time that both assets A and B have defaulted. A pool of N assets with loan-to-value ratios chosen independently and identically distributed using the distribution

$$\min(0.750 + 0.128 \times \text{Uniform}(0, 1), 1)$$

The simulation is currently based on using loan-to-value, largely because the report from H.U.D provided real data from which to fit the hazard function. It is certainly the case that more than loan-to-value may be relevant. As noted, geographical data, such as zip codes, may be extremely useful. In particular, two mortgages with respective loan-to-value ratios should be more correlated if the assets are geographically within the "same" real estate market (*i.e.* same city or metro-area) than if they are separated. Geographic seperation does not imply that the assets decorrelate, because other factors may contribute to asset correlation. Our thinking is that conditioned on other factors, loan-to-value becomes the final determining factor in asset correlation and in the absence of real market data, we would rather not speculate any further than necessary. Regardless, how correlated two assets "truly" are is of importance and errors in determining the true asset correlations contribute and increase the evalution of $\mathcal{A}(t)$.

Given two assets, A and B, the choice of covariance function used was $\cos(\frac{\pi}{2}(l_A - l_B))$, where l_A and l_B are the initial loan-to-value ratios for assets A and B. The matrix time-series $\delta_{AB}(t)$ may now be computed from the matrix of joint default times and subsequently, the time-series $\mathcal{A}(t)$ may then be computed.

There are several empirical observations concerning $\mathcal{A}(t)$. Although it is clear that $\mathcal{A}(t)$ is random and when $\delta_{AB}(t)$ is not identically 1, for all pairs, $\mathcal{A}(t)$ tends to converge to α.

1. The rate of convergence at which $\mathcal{A}(t) \to \alpha$ appears to depend on α.
2. Determining if $\mathcal{A}(t)$ has converged is subtle, because by the time one is confident about the estimate of α, t (time) may be unacceptably large.
3. It is the path properties of $\mathcal{A}(t)$ that are crucial.
4. The regularity of $\mathcal{A}(t)$ is also a function of number of assets N.

The important question concerns short time horizons and determining whether an acceptable estimate of α has been computed. If no contagion exists, then although $\mathcal{A}(t)$ will fluctuate, it should, in general, remain near 0, but having a point estimate of $\mathcal{A}(t)$ near 0 is not necessarily an indicator of convergence, because **it is the trend of over intervals of time that is really important and whether the variance of $\mathcal{A}(t)$ is suitably small, which is again a function of time**.

7 Running Statistic

$\mathcal{A}(t)$ "counts" the degree to which defaults that have occurred differ from the probabilities from a parameterized distribution and which value of this parameter minimizes this discrepancy. The rate of change of this statistic is important. To make matters more complex, if α is relatively small then although contagion still exists it can remain under the radar for quite awhile. If α is relatively large, then although it is above the radar, there is considerably less time to make decisions concerning the pool of assets. The situation is more stable for larger asset pools, but the behavior of $\mathcal{A}(t)$ over short time horizons is still the crux of the matter.

In order to get a sense of how this process behaves, 64 realizations of the process described above were simulated and the values of $\mathcal{A}(t)$, the percentage change of $\mathcal{A}(t)$, and the percentage change of defaults occurred were gathered and plotted. Each realization was run until no more than 7 percent of all possible pairs had jointly defaulted, which for these choices of parameters occurs at approximately $t = 1$ or one year.

Apparent from Figure 6 is that even for synthesized data using $\alpha = 2.5$ the variance of $\mathcal{A}(t)$ is initially quite large. Although it is a matter of opinion as to when the variance has suitably decreased, it is fair to say that between $t = 0.75$ and $t = 1$ the variance is acceptable and the joint defaults still have not exceeded 7 percent.

At least for $\alpha = 2.5$, and the functional dependency for the contagion effect, there is a time interval during which the contagion is ramping up and its effects have yet to propagate. Nominally this is between 9 months to a year after which these mortgages were initiated. Furthermore, under these assumptions there is nominally half a year before the contagion really sets in. Table 3 gives $\mathcal{A}(t)$ statistics, for the indicated choices of parameters. For viewing, figure 9 and figure 10 respectively graph the log-variance and variance of $\mathcal{A}(t)$.

Table 3. Statistics for $\alpha = 2.5, N = 100$

$\mathcal{A}(t)$ statistics for $\alpha = 2.5, N = 100$						
	Time	Mean	Variance	Log Variance	Skew	Kurtosis
1	0.0833	6.2095	53.262	3.9752	0.7680	-0.6825
2	0.1250	4.2722	20.205	3.0059	0.6980	-0.5935
3	0.1667	3.2992	9.6242	2.2643	0.3906	-1.1298
4	0.2083	2.8603	5.3543	1.6779	0.2128	-0.9491
5	0.2500	2.6651	3.4780	1.2465	0.1407	-0.8028
6	0.2917	2.5310	2.4356	0.8902	-0.0570	-1.0139
7	0.3333	2.4302	1.6834	0.5208	-0.0895	-0.8322
8	0.3750	2.4373	1.0255	0.0252	-0.3132	-0.5219
9	0.4167	2.4476	0.7263	-0.3198	-0.2917	0.0819
10	0.4583	2.4762	0.4472	-0.8047	-0.5961	1.4491
11	0.5000	2.5040	0.3193	-1.1416	-0.5269	0.7446
12	0.5417	2.4865	0.2394	-1.4296	-0.4059	0.3112
13	0.5833	2.4667	0.1813	-1.7076	-0.5756	0.4966
14	0.6250	2.4849	0.1294	-2.0448	-0.6061	0.4958
15	0.6667	2.4905	0.1014	-2.2887	-0.8166	1.4329
16	0.7083	2.5024	0.0789	-2.5408	-0.8704	2.0296
17	0.7500	2.5040	0.0561	-2.8806	-0.8924	1.7543
18	0.7917	2.5016	0.0465	-3.0683	-0.9999	1.7938
19	0.8333	2.5190	0.0319	-3.4451	-0.5712	0.2092
20	0.8750	2.5317	0.0220	-3.8167	-0.4862	0.0899
21	0.9167	2.5294	0.0163	-4.1166	-0.3639	-0.1875

8 Some Formalism

For each choice of ρ_{AB} and choice of hazard function, a probability measure is being induced with distribution function:

$$\mu_{AB}((0,t)) = P(\tau_A < t, \tau_B < t) = \Phi_2(\Phi^{-1}(P(\tau_A < t)), \Phi^{-1}(P(\tau_B < t)), \rho_{AB}).$$

Define a measure-valued matrix, over all pairs A and B, to be M, where the index set is denoted by $\{1, 2, \cdots, N\}$ over the set of assets.

$$M = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1N} \\ \vdots & \ddots & \vdots \\ \mu_{N1} & \cdots & \mu_{NN} \end{pmatrix}$$

The contagion term perturbation also induces a measure-valued matrix.

$$M^{\alpha} = \begin{pmatrix} \mu^{\alpha}_{11} & \cdots & \mu^{\alpha}_{1N} \\ \vdots & \ddots & \vdots \\ \mu^{\alpha}_{N1} & \cdots & \mu^{\alpha}_{NN} \end{pmatrix}$$

Furthermore, consider the matrix that represents the pair-wise defaults.

$$\Delta(t) = \begin{pmatrix} \delta_{11}(t) & \cdots & \delta_{1N}(t) \\ \vdots & \ddots & \vdots \\ \delta_{N1}(t) & \cdots & \delta_{NN}(t) \end{pmatrix}$$

Recall that the δ's are 0 or 1 determined by whether pairs of assets have both defaulted by time t. Using a standard sum of squares norm, define

$$\|\Delta(t) - M^{\alpha}((0,t))\|^2 \equiv \sum_{AB} (\delta_{AB}(t) - p^{\alpha}_{AB})^2$$

Now, we have given by

$$\mathcal{A}(t) = \min_{\alpha > 0} \|\Delta(t) - M^{\alpha}((0,t))\|^2$$

We are looking for the nearest M^{α} to $\Delta(t)$ with respect to $\|\cdot\|$, and $\mathcal{A}(t)$ represents the best α at time t. The functional form for the perturbations were chosen because contagion is connected to branching processes, but we chose the simple type of branching. This could have been made more complex. Figure 2 depicts a measure of a perturbation's distance from the distribution of pair-wise defaults, and does not actually imply the geometry.

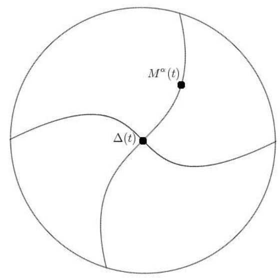

Fig. 2. Depiction of measure space perturbation

9 Observations, conclusions, and next steps

The approach taken has been straight-forward: we have postulated that the hazard function associated with the standard copula approach may be perturbed by an exponential functional corresponding to Malthusian growth. Although the form of this perturbation has not been rigorously established, its effect is not unreasonable. Regardless, one may choose to consider other perturbations. Once the perturbation term is chosen, the method for fitting is really a regression or curve-fitting type of analysis.

The form of the perturbation was chosen because it leads to a simple altering of the tail distribution of hazard function and results in a phase transition for the copula method. To be more precise, the phase transition is associated with the branching process, and for $\alpha > 0$ is the phase for which the branching process has positive probability of spreading and not dying out.

In the $\alpha > 0$ case, there is a period of time before which the effects of the contagion phase are negligible. **After an incubation period,**

the effects of the contagion phase become macroscopically noticeable on the magnitude of the pool of assets, but as the simulation shows there is an interval of time in which the rate could be estimated. As time elapses, the variance decreases, as does the window of time before the macroscopic effects of contagion become unavoidable. There is a trade-off between certainty about the estimates and the time frame remaining in which to make decisions. The reader should take note that for very short-time asymptotics the variance in the estimates are "large", but this balanced by the number of defaults being small. As the time evolves, the number of defaults increases and the law large numbers starts to take hold and variance decreases.

Of course, the real world events would be more complex than the synthesized events and in fact, time delays would play a significant role in altering the variance in estimating α, but the methods are still relevant as a starting point for future advances. The choice of functions which may be used for perturbation of the hazard function should be further explored, as well as refining the dependency on other relevant data beyond loan-to-value, such as geographical and other asset attributes. For example, rather than using a Malthusian growth behavior, a logistics growth term could have been used and although estimates would be different, the framework would stay the same. The exponential growth is curtailed with logistics growth, but initially remains exponential. The long term/tail behavior of a purely exponential correction may turn out to be extreme and overly conservative, but this is somewhat a moot point when considering current events in the mortgage markets.

Ultimately, use of real data is needed to determine the correct perturbation of the hazard function, refinements of the hazard functions to attributes, and the realistic correlation functions. Although some real data has been involved in this work, more real data would have been helpful in calibrating the approach, whereby making the results more useful to the practitioner.

9.1 Over-estimating Defaulting

Although this case seems pedantic when considering the current mortgage crisis, note that the initial estimate of hazard function and correlations can result in a situation in which a positive perturbation term only increases error. This situation can arise if the initial probability measure over-estimates the defaulting. In this case, a negative perturbation term is needed in order to reduce the error. Such a perturbation can be an exponential correction or another function. Although this is not realistic for the current events, it is possible and should be considered when considering "long" positions or a bubble that is expanding. In practice, one would consider an expansion of the market, followed by a contraction.

9.2 Time-varying loan-to-value

In this paper, loan-to-value has been treated as a static variable established when a mortgage was initiated, but this is not realistic. Fluctuating, albeit currently mostly decreasing, housing pricing causes the loan-to-value ratio to flucuate and in some cases the value can move under the loan resulting in a loan-to-value ≥ 1. In addition, mortgage payments will also adjust the loan-to-value ratios. Formulas discussed need to be adjusted for these cases, but the principle is to measure "skin in the game" and the less "skin" in the game a borrower has, the more likely the loan is to default. Having loan-to-value greater than or equal 1 does not necessarily imply instant defaulting, but it certainly represents an extremely risky position.

The devaluation of assets as been a significant contributing force resulting in the spread of contagion in the mortgage market. It is conceivable that if one, in hindsight, adjusts the loan-to-value ratios for the value decline, then the resulting alpha might not be very large. The point is to estimate alpha in foresight and not hindsight. Of course, the situation is one of evaluating and updating in an ongoing process, with the objective to predict the extent of defaulting.

The need for α is a result of having to correct for error in initial estimates.

10 Acknowlegments

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3. T. E. Harris, *The Theory of Branching Processes*, Prentice-Hall, 1963

11 Figures

Note: the legends within the figures are likely to require enlargement using electronic viewing in order to read.

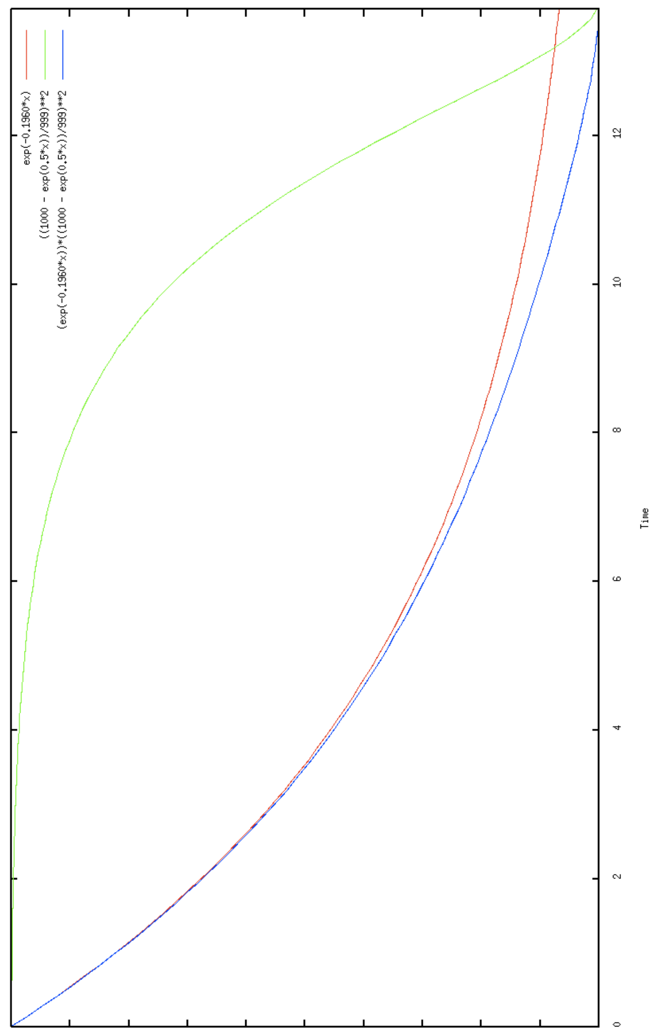


Fig. 3. $CONTAG(1000, 0.5, t)$

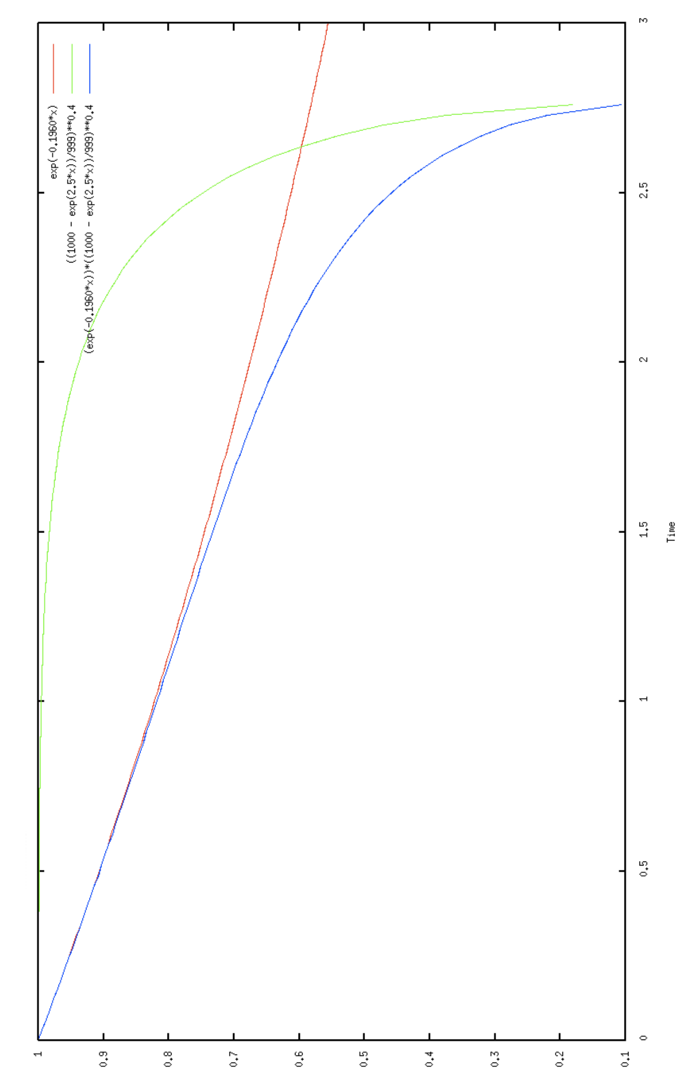


Fig. 4. $CONTAG(1000, 2.5, t)$

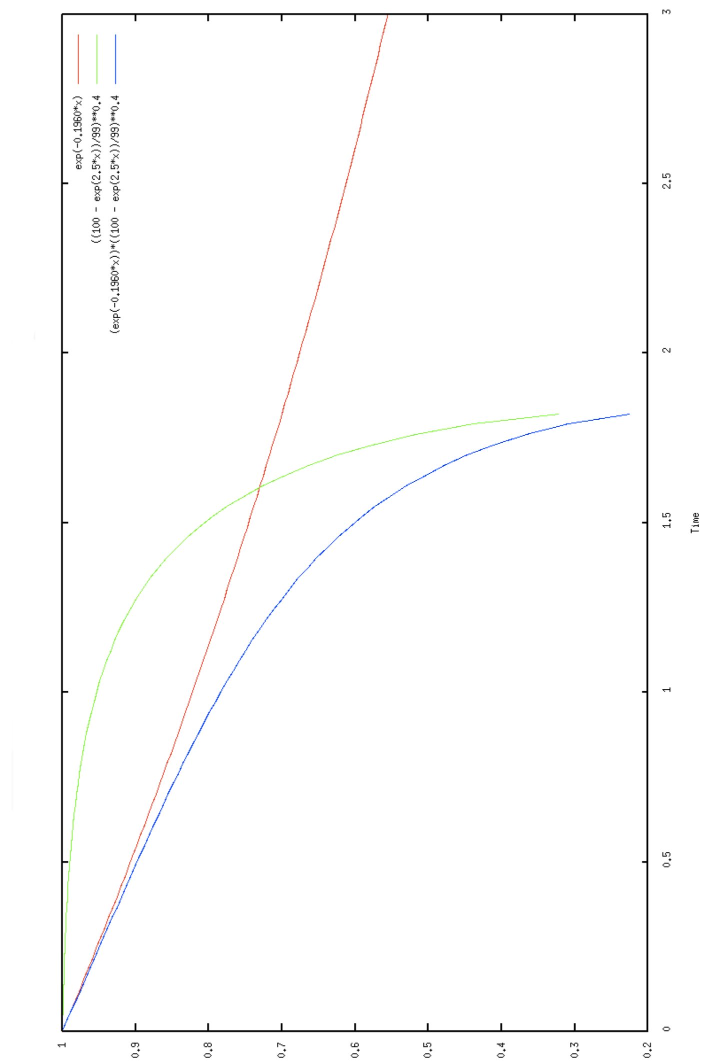


Fig. 5. $CONTAG(100, 2.5, t)$

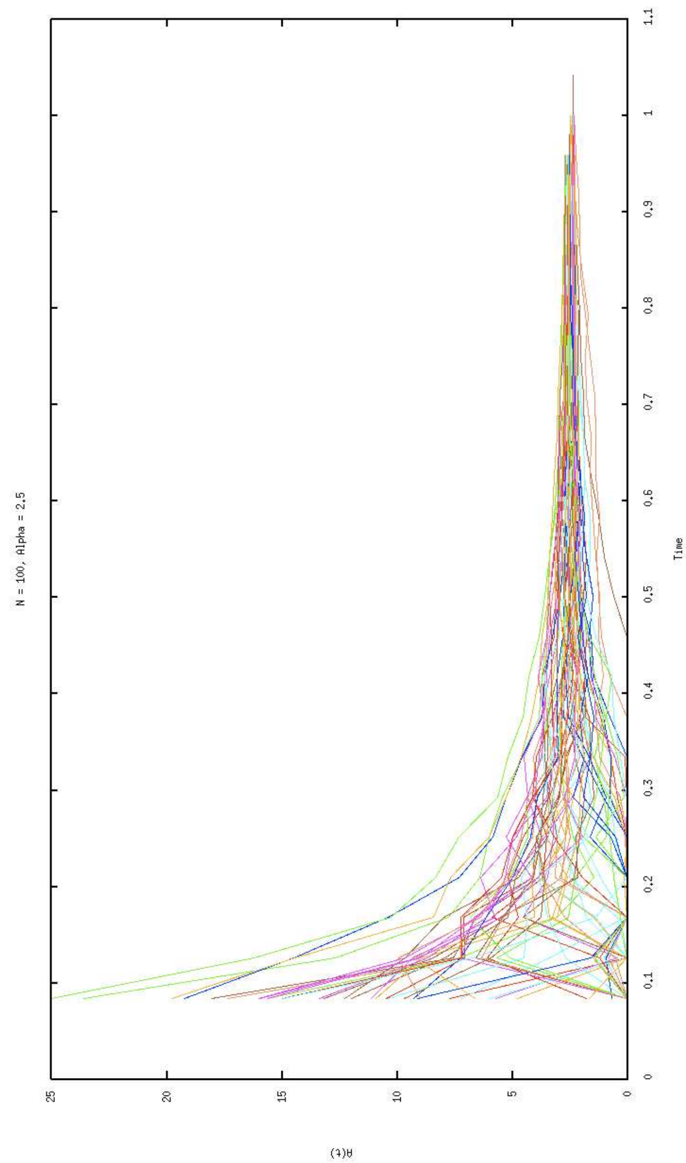


Fig. 6. realizations of $\mathcal{A}(t)$

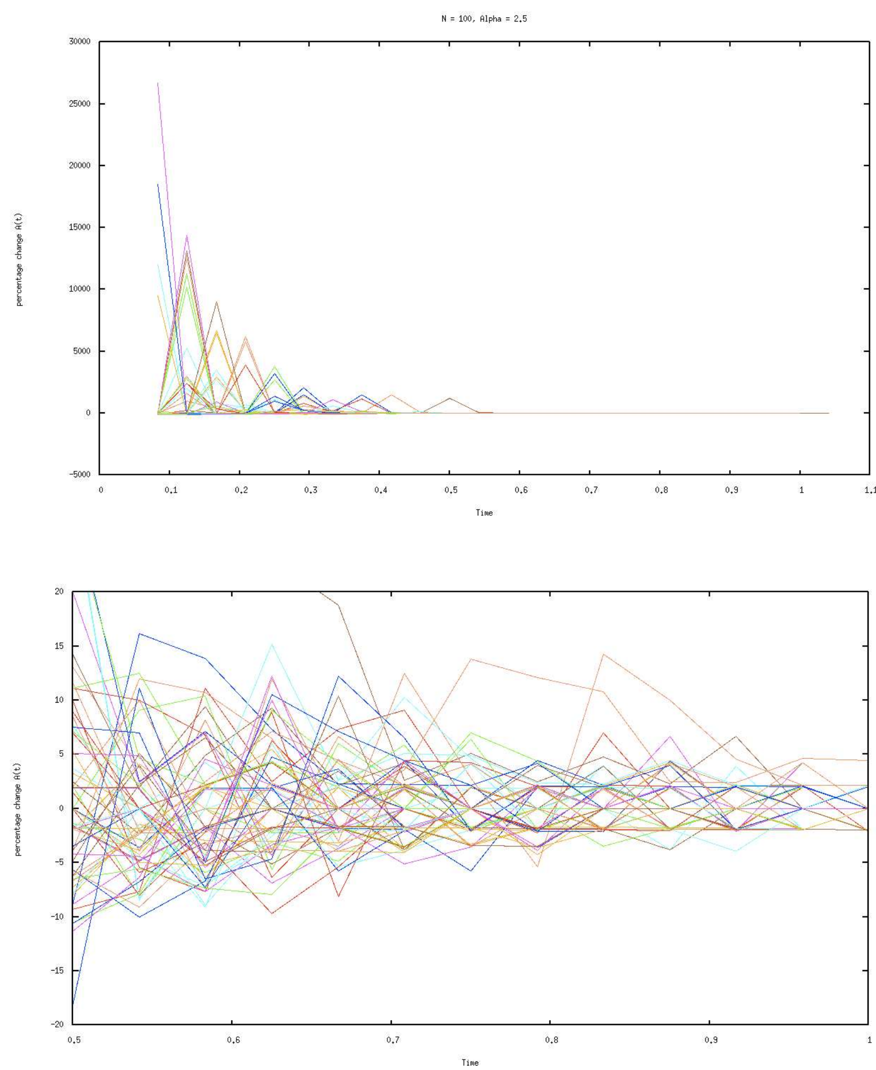
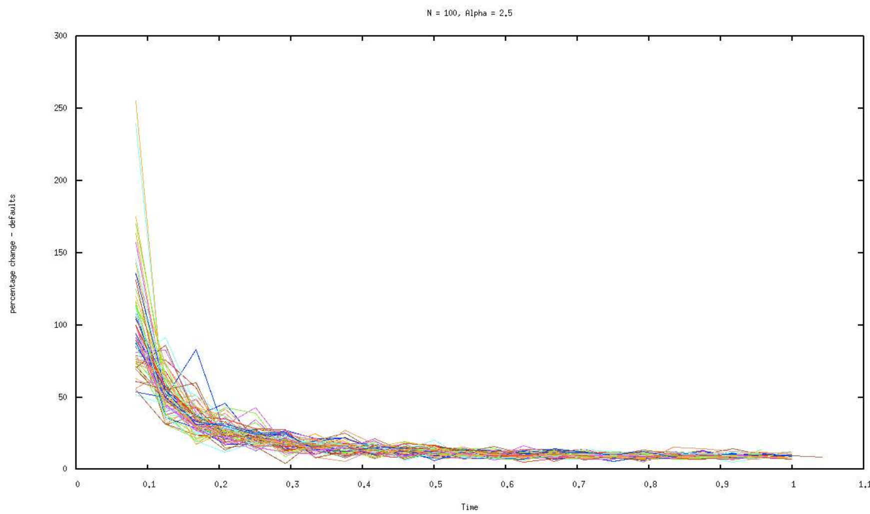


Fig. 7. Percentage Change of $\mathcal{A}(t)$ (Second figure is a subset of the first)

The percentage change of quantities $\mathcal{A}(t_2)$ and $\mathcal{A}(t_1)$, with times $t_2 > t_1$ is

$$(\frac{(\mathcal{A}(t_2) - \mathcal{A}(t_1))}{\mathcal{A}(t_1)}) \times 100$$

.



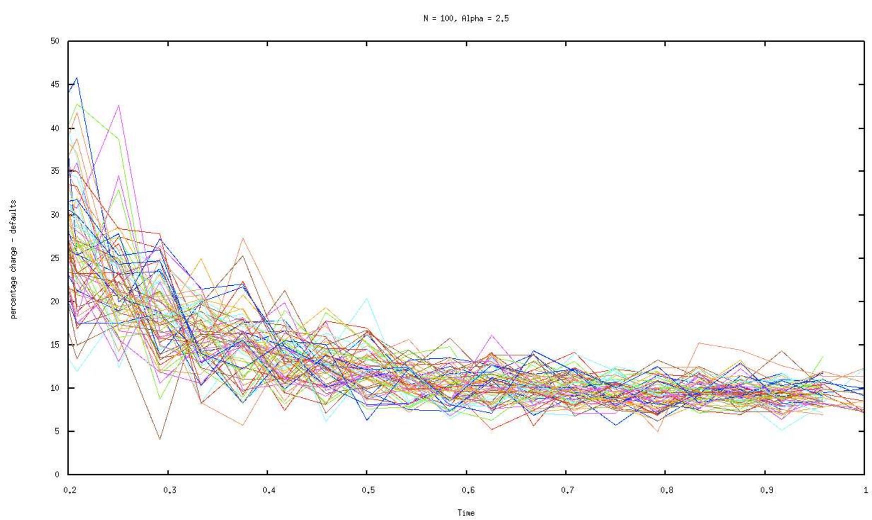


Fig. 8. Percentage Change of Defaults (Second figure is a subset of the first)

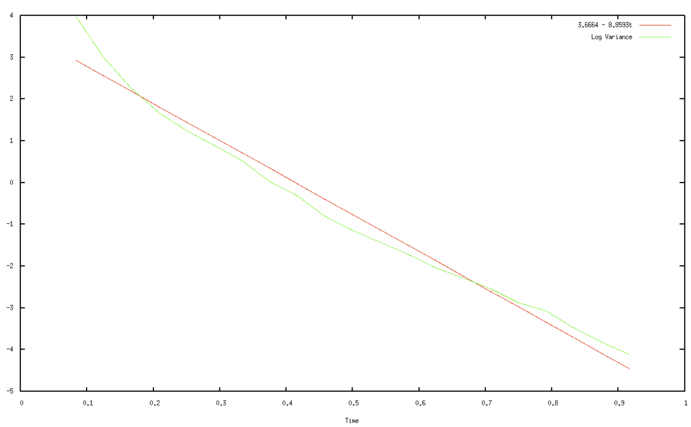


Fig. 9. log variance $\mathcal{A}(t)$

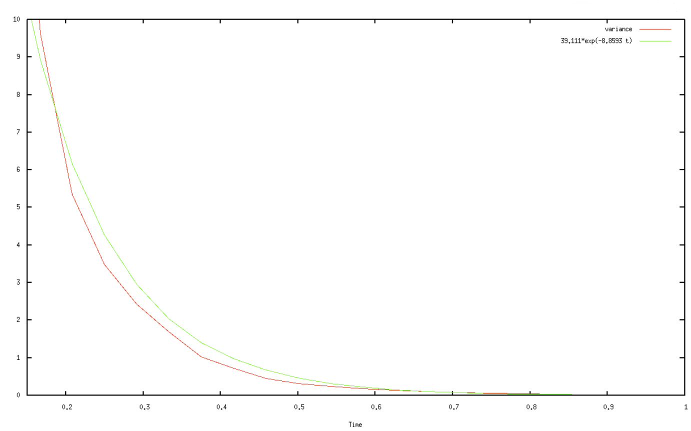


Fig. 10. variance $\mathcal{A}(t)$