Efficient Capital Allocation through Optimization

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Abstract

In this paper, we formulate the Capital Allocation problem as an optimization problem in which we seek the mix of business that maximizes an insurance company's *Expected Net After Tax Income* subject to a constraint on the *Tail Value at Risk (TVAR)*. Using the method of Lagrange multipliers, we demonstrate that the returns on the respective TVAR contributions, so-called RORAC, are equal across all lines of business when the mix of business is optimal. We refer to this state as RORAC Equilibrium. We then investigate the impact on RORAC Equilibrium of introducing premium constraints in the optimization problem. We show that these constraints impose a cost on the company's Net After Tax Income. When the line of business returns are adjusted for the applicable costs, equilibrium is maintained. Using commercially available optimization software, we solve the optimization problem for a fictitious start-up company and we show several points on the so-called efficient frontier curve of the company. Cases with various premium constraints are also examined. Although the discussions in this paper center on the TVAR, the conclusions hold true for any conditional expected value measure.

Keywords: Optimization; Lagrange Function; Lagrange Multipliers; Capital Allocation; RORAC Equilibrium.

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Introduction

Capital Allocation remains one of the most intriguing and perhaps most controversial topics in the Casualty Actuarial literature and, indeed, in the financial literature. The recent crisis in the financial industry has put the spotlight on Enterprise Risk Management and has added vigor to the debate on Capital Allocation. Actuarial thinking is divided into two firmly entrenched camps: those who believe that Capital can be allocated and those who don't believe that it can.

But what is Capital Allocation and why is it relevant? Actuaries on both sides of the allocation debate have implicitly defined Capital Allocation as the decomposition or division of "Capital" into pieces that can be assigned to business units, lines of business, or even individual contracts. The goals of this decomposition – Venter [7] reminds us – "include testing the profitability of business units and determining which units could best be grown to add value to the firm." While the explicit decomposition of "Capital" might well be one of many available processes that companies rely on in order to make allocation decisions, this decomposition process itself need not define Capital Allocation. In this paper, we define Capital Allocation more generically as:

> *…[any] process of how businesses divide their financial resources and other sources of capital to different processes, people and projects. Overall, it is management's goal to optimize capital allocation so that it generates as much wealth as possible for its shareholders.* Investopedia [3]

In a generic sense, all insurance companies allocate their Capital. By virtue of being in business, insurance companies routinely decide between expanding into or retreating from a territory, launching a new product or discontinuing an underperforming one, or simply renewing or canceling an insurance contract. All of these are Capital Allocation decisions, few of which, we suspect, ever involve the decomposition of Capital. The question should not so much be whether companies allocate Capital since they all do. Rather the question should be whether a company's allocation of Capital is efficient. Capital Allocation is efficient when it results in the greatest return for a given amount of risk or, alternatively, when it results in the lowest amount of risk for a given return.

In this paper, we endeavor to find an efficient allocation of Capital: that is to find a mix of business that maximizes shareholder wealth subject to various risk and premium constraints that may be imposed by the shareholders themselves, regulators, rating agencies, or the marketplace. This is a constrained optimization problem similar, for instance, to that confronted by an industrial company that has to choose the mix of production methods that maximizes income while facing limits on its pollution emission.

The remainder of this paper is organized as follows: In section 1, we establish some definitions. In section 2, we formulate the optimization problem without any premium constraints – except that premiums are assumed to be positive. Using the Lagrange function and Lagrange multipliers, we demonstrate the principal result of this paper: **that RORAC is equal across all lines of business when the mix of business is optimal.** In section 3, we add premium constraints to the basic optimization problem. Again, using the method of Lagrange multipliers, we show that when RORAC is adjusted for the costs imposed by the premium constraints, equilibrium is maintained. In section 4, we solve the optimization problem for a fictitious start-up company. We show several points on the so-called efficient frontier curve of the company and we offer some observations.

Section 1 – Definitions

For all the debate about Capital Allocation, it is not always clear which Capital is being discussed or being allocated. The term Capital is used in at least two different contexts. On the one hand, Capital is understood simply as a measure of value, wealth, or the sum of all the current and future resources – financial, technical, technological, intellectual or other – available to a firm. We refer to this as **Real Capital**. On the other hand, the term Capital is used as a measure of the risk undertaken by the firm. Examples of this measure include Variance, Value at Risk, Tail Value at Risk, or Expected Policyholder Deficit. We refer to this as **Risk Capital**. Although these two notions of Capital intersect quite frequently, they are very distinct. Real Capital is the reason companies are in business whereas Risk Capital is a constraint on business. Companies look to grow or maximize Real Capital while they seek to limit or minimize Risk Capital. Real Capital is a concrete accounting or financial measure defined by a relatively narrow set of rules whereas Risk Capital is in principle an abstract theoretical measure belonging to the realm of Statistics and Actuarial Science. In this paper, we optimize Real Capital – or more precisely the change in Real Capital – subject to constraints on Risk Capital. We optimize Real Capital by making decisions about the lines of business, contracts, or territories to which a company's resources should be devoted. As mentioned above, we think broadly of the company's Real Capital as the aggregate of all its resources including its financial assets, its physical locations, its computer software and hardware, its underwriters, accountants, claim examiners, and, of course, actuaries.

In this paper, we measure Real Capital as the present value of assets minus the present value of liabilities. The change in Real Capital, as we have defined it, is represented by the **Discounted Net After Tax Income**. In the remainder of this paper, we will refer to this as **Net Income** for short. Risk Capital is measured by the **Tail Value at Risk (TVAR) of Net Income**. Other measures used in the paper are defined as follows:

RAROC = Expected Net Income ÷ Real Capital
RORAC = Expected Net Income ÷ Risk Capital
RORAC $_{\text{LOB } i}$ = Expected Net Income Contribution $_{\text{LOB } i}$ ÷ Risk Capital Contribution $_{\text{LOB } i}$ [1]
Risk Leverage = Risk Capital ÷ Real Capital
Premium Leverage = Premium ÷ Real Capital

[1] Net Income Contribution and Risk Capital Contribution for a line of business are defined later in the paper.

Section 2 - Mathematical Formulation of the basic Optimization Problem

Assume a company has access to two lines of business[2] and let p_1 and p_2 denote the premiums written in each line, respectively. Net Income is written as a linear function of p_1 and p_2 as:

$$\textit{Net Income} = p_1 r_1 + p_2 r_2 + B$$

where r_1 and r_2 are random variables representing the returns associated with p_1 and p_2, respectively, and B is a function of random variables representing the balance of the Net Income equation which does not depend on p_1 and p_2. Let $f(r_1, r_2)$ represent the joint distribution[3] of the random variables r_1 and r_2. Let $TVAR_\beta$ represent the β^{th} percentile tail value at risk of Net Income. We want to maximize the Expected Value of Net Income subject to a constraint on the Tail Value at Risk of Net Income.

We seek p_1 and p_2 that:

Maximize: $$p_1 E(r_1) + p_2 E(r_2) + E(B) \tag{2.1}$$

subject to: $$\frac{1}{\beta} \iint_C \left(p_1 r_1 + p_2 r_2 + B\right) f(r_1, r_2)\, dr_1 dr_2 \geq L_0 \text{ [4]} \tag{2.2}$$

where $C = \{(p_1, p_2, r_1, r_2, B)\ s.t.\ p_1 r_1 + p_2 r_2 + B \geq VAR_\beta\}$, VAR_β represents the β^{th} percentile Value at Risk, and L_0 represents the constraint on the TVAR. Also, we assume the premiums are non-negative so that:

$$p_1 \geq 0 \tag{2.3}$$

$$p_2 \geq 0 \tag{2.4}$$

We rewrite the optimization problem in standard form[5] as:

Maximize: $$p_1 E(r_1) + p_2 E(r_2) + E(B) \tag{2.1'}$$

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[2] The problem can be easily extended to *n* lines of business. Also, we can substitute individual contracts for lines of business.

[3] We assume that this distribution does not depend on the values of p_1 and p_2. This is consistent with a competitive market assumption in which no one company can influence price levels. It is possible that certain niches of the insurance market may in fact violate that assumption. Certainly, one should be able to adjust the Net Income formula to reflect a Combined Ratio distribution that varies with the premium amount. The optimization problem probably becomes much thornier

[4] Alternatively, for discrete distributions $TVAR_\beta = \frac{1}{\beta} \sum_C (p_1 r_1 + p_2 r_2 + B)\, p(r_1, r_2)$

[5] The optimization problem in standard form is typically written as:

Minimize (Maximize) $\quad f_0(x)$

subject to $\quad f_i(x) \leq 0, \quad i = 1, \cdots, m$

$\quad h_i(x) = 0, \quad i = 1, \cdots, p$

subject to: $$-\frac{1}{\beta}\iint_C (p_1 r_1 + p_2 r_2 + B) f(r_1, r_2) dr_1 dr_2 \le -L_0 \qquad (2.2')$$

$$-p_1 \le 0 \qquad (2.3')$$

$$-p_2 \le 0 \qquad (2.4')$$

We write the Lagrange function[6] of the Optimization problem as:

$$\Lambda(p_1, p_2, \lambda_0, \lambda_1, \lambda_2) = p_1 E(r_1) + p_2 E(r_2) + E(B)$$
$$-\lambda_0 \left[\frac{1}{\beta}\iint_C (p_1 r_1 + p_2 r_2 + B) f(r_1, r_2) dr_1 dr_2 + L_0 \right] - \lambda_1 p_1 - \lambda_2 p_2 \qquad (2.5)$$

Taking partial derivatives[7] of the Lagrange function with respect to p_1 and p_2 yields:

$$\frac{\partial \Lambda}{\partial p_1} = E(r_1) - \lambda_0 \left[\frac{1}{\beta}\iint_C r_1 f(r_1, r_2) dr_1 dr_2 \right] - \lambda_1 \qquad (2.6)$$

$$\frac{\partial \Lambda}{\partial p_2} = E(r_2) - \lambda_0 \left[\frac{1}{\beta}\iint_C r_2 f(r_1, r_2) dr_1 dr_2 \right] - \lambda_2 \qquad (2.7)$$

Assuming $p_1 > 0$ and $p_2 > 0$ (otherwise, the problem is trivial), then $\lambda_1 = \lambda_2 = 0$

Setting (2.6) and (2.7) equal to zero yields:

$$\lambda_0^* = \frac{E(r_1)}{\frac{1}{\beta}\iint_C r_1 f(r_1, r_2) dr_1 dr_2} \qquad (2.8)$$

$$\lambda_0^* = \frac{E(r_2)}{\frac{1}{\beta}\iint_C r_2 f(r_1, r_2) dr_1 dr_2} \qquad (2.9)$$

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[6] Using the notations from footnote 5, the Lagrange function is given by:

$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$. See S. Boyd [2] for a complete treatment of Lagrange functions.

[7] Equations (2.6) and (2.7) rely on the following equality:

$$\frac{\partial}{\partial p_i}\frac{1}{\beta}\iint_C (p_1 r_1 + p_2 r_2 + B) f(r_1, r_2) dr_1 dr_2 = \frac{1}{\beta}\iint_C \frac{\partial}{\partial p_i}(p_1 r_1 + p_2 r_2 + B) f(r_1, r_2) dr_1 dr_2; \quad i = 1,2.$$

This equality is not trivial as the region C varies with p_1 and p_2. It is demonstrated in Major [4] where the gradient of the TVAR is derived by applying the “Integral over the Surface” formula. The latter formula is described and proven in Uryasev [9].

Then, we obtain from equations (2.8) and (2.9):

$$\frac{E(r_1)}{\frac{1}{\beta}\iint\limits_C r_1 f(r_1,r_2)dr_1dr_2} = \frac{E(r_2)}{\frac{1}{\beta}\iint\limits_C r_2 f(r_1,r_2)dr_1dr_2} \qquad (2.10)$$

Let p_1^* and p_2^* represent the solution to the optimization problem. If we multiply both the numerator and the denominator of the left and right sides of equation (2.10) by p_1^* and p_2^*, respectively, we obtain our main result:

$$\frac{p_1^*E(r_1)}{\frac{1}{\beta}\iint\limits_C p_1^* r_1 f(r_1,r_2)dr_1dr_2} = \frac{p_2^*E(r_2)}{\frac{1}{\beta}\iint\limits_C p_2^* r_2 f(r_1,r_2)dr_1dr_2} \qquad (2.11)$$

where $\frac{1}{\beta}\iint\limits_C p_1^* r_1 f(r_1,r_2)dr_1dr_2$, $\frac{1}{\beta}\iint\limits_C p_2^* r_2 f(r_1,r_2)dr_1dr_2$, and $p_1^*E(r_1)$, $p_2^*E(r_2)$ represent the respective **Tail Value at Risk Contributions (or Risk Capital Contributions)[8]** and respective **Expected Net Income Contributions** of lines 1 and 2. We will refer to (2.11) as the RORAC Equilibrium Equation.

Section 3 - Mathematical Formulation of the Optimization Problem with Premium Constraints

Let's now introduce premium constraints to the original optimization problem.

We seek p_1 and p_2 that:

Maximize: $\quad p_1E(r_1) + p_2E(r_2) + E(B) \qquad (3.1)$

Subject to: $\quad \frac{1}{\beta}\iint\limits_C \left(p_1r_1 + p_2r_2 + B\right) f(r_1,r_2)dr_1dr_2 \ge L_0 \qquad (3.2)$

$p_1 \ge 0 \qquad (3.3)$

$p_2 \ge 0 \qquad (3.4)$

$p_1 \le l_1 \qquad (3.5)$

$p_2 \le l_2 \qquad (3.6)$

$p_1 + p_2 \le l_3 \qquad (3.7)$

We rewrite the optimization problem in standard form as:

Maximize: $\quad p_1E(r_1) + p_2E(r_2) + E(B) \qquad (3.1')$

[8] See Venter [8] for some background on the notion of Risk Contributions.

subject to: $$-\frac{1}{\beta}\iint_C \left(p_1 r_1 + p_2 r_2 + B\right) f(r_1, r_2) dr_1 dr_2 \le -L_0 \qquad (3.2')$$

$$-p_1 \le 0 \qquad (3.3')$$

$$-p_2 \le 0 \qquad (3.4')$$

$$p_1 \le l_1 \qquad (3.5')$$

$$p_2 \le l_2 \qquad (3.6')$$

$$p_1 + p_2 \le l_3 \qquad (3.7')$$

We write the Lagrange function of the Optimization problem as:

$$\begin{aligned}
&\Lambda(p_1, p_2, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = p_1 E(r_1) + p_2 E(r_2) + E(B) \\
&-\lambda_0 \left[\frac{1}{\beta}\iint_C \left(p_1 r_1 + p_2 r_2 + B\right) f(r_1, r_2) dr_1 dr_2 + L_0 \right] \\
&-\lambda_1 p_1 - \lambda_2 p_2 + \lambda_3 (p_1 - l_1) + \lambda_4 (p_2 - l_2) + \lambda_5 (p_1 + p_2 - l_3) \qquad (3.8)
\end{aligned}$$

Taking partial derivatives of the Lagrange function with respect to p_1 and p_2 yields:

$$\frac{\partial \Lambda}{\partial p_1} = E(r_1) - \lambda_0 \left[\frac{1}{\beta}\iint_C r_1 f(r_1, r_2) dr_1 dr_2 \right] - \lambda_1 + \lambda_3 + \lambda_5 \qquad (3.9)$$

$$\frac{\partial \Lambda}{\partial p_2} = E(r_2) - \lambda_0 \left[\frac{1}{\beta}\iint_C r_2 f(r_1, r_2) dr_1 dr_2 \right] - \lambda_2 + \lambda_4 + \lambda_5 \qquad (3.10)$$

Assuming $p_1 > 0$ and $p_2 > 0$ (otherwise, the problem is trivial), then $\lambda_1 = \lambda_2 = 0$
Setting (3.9) and (3.10) equal to zero yields:

$$\lambda_0^* = \frac{E(r_1) + \lambda_3^* + \lambda_5^*}{\frac{1}{\beta}\iint_C r_1 f(r_1, r_2) dr_1 dr_2} \qquad (3.11)$$

$$\lambda_0^* = \frac{E(r_2) + \lambda_4^* + \lambda_5^*}{\frac{1}{\beta}\iint_C r_2 f(r_1, r_2) dr_1 dr_2} \qquad (3.12)$$

Then, we obtain from equations (3.11) and (3.12):

$$\frac{E(r_1) + \lambda_3^* + \lambda_5^*}{\frac{1}{\beta}\iint_C r_1 f(r_1, r_2) dr_1 dr_2} = \frac{E(r_2) + \lambda_4^* + \lambda_5^*}{\frac{1}{\beta}\iint_C r_2 f(r_1, r_2) dr_1 dr_2} \qquad (3.13)$$

Let p_1^* and p_2^* represent the solution to the optimization problem. If we multiply both the numerator and the denominator of the left and right sides of equation (3.13) by p_1^* and p_2^*, respectively, we obtain the following:

$$\frac{p_1^*(E(r_1)+\lambda_3^*+\lambda_5^*)}{\frac{1}{\beta}\iint_C p_1^* r_1 f(r_1,r_2)dr_1dr_2} = \frac{p_2^*(E(r_2)+\lambda_4^*+\lambda_5^*)}{\frac{1}{\beta}\iint_C p_2^* r_2 f(r_1,r_2)dr_1dr_2} \tag{3.14}$$

where $\frac{1}{\beta}\iint_C p_1^* r_1 f(r_1,r_2)dr_1dr_2$ and $\frac{1}{\beta}\iint_C p_2^* r_2 f(r_1,r_2)dr_1dr_2$ represent the **Tail Value at Risk Contributions** of lines 1 and 2, respectively. We refer to the left and right sides of equation (3.14) as the **Adjusted RORAC** for lines 1 and 2, respectively. We refer to (3.14) as the Adjusted RORAC Equilibrium Equation.

<u>Interpretation of the Lagrange Multipliers</u>

The Lagrange multipliers lend themselves to some interesting interpretations. Taking the partial derivatives of the Net Income function with respect to the constraint variables leads to the following equations:

$$\lambda_0^* = \frac{\partial(Net\ Income)}{\partial L_0} \tag{3.15}$$

$$\lambda_3^* = -\frac{\partial(Net\ Income)}{\partial L_3} \tag{3.16}$$

$$\lambda_4^* = -\frac{\partial(Net\ Income)}{\partial L_4} \tag{3.17}$$

$$\lambda_5^* = -\frac{\partial(Net\ Income)}{\partial L_5} \tag{3.18}$$

λ_0^* can be interpreted as the additional income that would be gained by relaxing the TVAR constraint (by one unit) when the portfolio is optimal. Similarly, λ_3^*, λ_4^*, and λ_5^* can be interpreted as the as the additional income that would be gained by relaxing the premium constraints L_3, L_4, and L_5 (by one unit), respectively, when the portfolio is optimal. Economists typically refer to this as the shadow price of the constraint or the most the firm would be willing to pay to relax the constraint. In the appendix, we derive equations (3.15) and (3.16).

Section 4 – Allocating Capital for a Start-Up

To illustrate these concepts, we will look to find the optimal business mix over a one-year time horizon[9] for a start-up insurance entity to which shareholders have contributed $250M of seed money. The entity has access to three lines of business: A, B, and C. We will generate 50,000 combined ratio scenarios based on the distributions given in the table below.

Line of Business	Comb Ratio Distribution	Mean	Standard Deviation
A	Lognormal	105%	20%
B	Lognormal	100%	32.5%
C	Lognormal	50%	40%

Also, lines A and B have a dependency relationship defined by a Clayton Copula[10] with $\theta = 2$. This implies a rank correlation of about .68 between these two lines. Line C is independent of both lines A and B. It is assumed that the Loss Ratio distributions do not change with the volume of business that is written by the entity. Premiums for all lines are paid at the beginning of the period. Losses for lines A, B, and C are paid at the end of years 4, 5, and 1, respectively. It is further assumed that the company's assets are invested in a risk free security yielding a fixed annual return of 2.5%.

The company's Gross Income is subject to a 20% tax. Let p_A, p_B, and p_C denote the premiums written in lines A, B, and C and q_A, q_B, and q_C, the combined ratios associated with these lines. The company's Net After Tax Income (in Millions) at the end of period 1 is given by:

$$Net\ Income = .8 \times \left[250 \times .025 + \left(p_A + p_B + p_C\right) \times 1.025 - \frac{\left(p_A q_A\right)}{1.025^3} - \frac{\left(p_B q_B\right)}{1.025^4} - p_C q_C \right] \text{ or}$$

$$Net\ Income = 5 + \left(.82 - \frac{.8 q_A}{1.025^3} \right) p_A + \left(.82 - \frac{.8 q_B}{1.025^4} \right) p_B + \left(.82 - .8 q_C\right) p_C$$

Let $r_A = .82 - \frac{.8 q_A}{1.025^3}$, $r_B = .82 - \frac{.8 q_B}{1.025^4}$, and $r_C = .82 - .8 q_C$, then

$$Net\ Income = 5 + r_A p_A + r_B p_B + r_C p_C$$

$$p(q_A, q_B, q_C) = p(r_A, r_B, r_C)$$

The company is subject to a constraint L_0 on the 1st Percentile Tail Value at Risk.

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[9] We are not recommending doing an allocation over a one-year time horizon. The one-year horizon is used for computational ease.

[10] The Clayton copula is given by: $C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1); \quad \theta \geq 0.$ See Trivedi and Zimmer [6] for a description of the properties of the Clayton Copula

Problem 1 – No premium Constraint

The problem is stated as follows:

Find p_A, p_B, and p_C that:

Maximize $Expected\ Net\ Income = 5 + E(r_A)p_A + E(r_B)p_B + E(r_C)p_C$

Subject to: $TVAR_{1\%} = \frac{1}{1\%}\sum_S [5 + r_A p_A + r_B p_B + r_C p_C] p(r_A, r_B, r_C) \geq L_0$

where $S = \{(p_1, p_2, p_3, r_1, r_2, r_3, B)\ s.t.\ Net\ Income \geq VAR_{1\%}\}$

$p_A \geq 0$
$p_B \geq 0$
$p_C \geq 0$

Using equation 2.5, we write the Lagrange function as

$$\Lambda(p_A, p_B, p_C \lambda_0, \lambda_1, \lambda_2, \lambda_3) = p_A E(r_A) + p_B E(r_B) + p_C E(r_C) + 5$$
$$-\lambda_0 \left[\frac{1}{1\%}\sum_S [r_A p_A + r_B p_B + r_C p_C + 5] p(r_A, r_B, r_C) + L_0 \right] - \lambda_1 p_A - \lambda_2 p_B - \lambda_3 p_C$$

The RORAC Equilibrium Equation (2.11) for Problem 1 is written as:

$$\frac{p_A^* E(r_A)}{\frac{1}{1\%}\sum_S r_A p_A p(r_A, r_B, r_C)} = \frac{p_B^* E(r_B)}{\frac{1}{1\%}\sum_S r_B p_B p(r_A, r_B, r_C)} = \frac{p_C^* E(r_C)}{\frac{1}{1\%}\sum_S r_C p_C p(r_A, r_B, r_C)}$$

The solutions to Problem 1 at various TVAR levels are shown in table 4.1. In table 4.2, we show results for various points below the efficient frontier. In Graph 4.1, we show both sets of points.

Problem 2 – Constraint: Total Premium for lines A, B, and C capped at 300M

The problem is stated as follows:

Find p_A, p_B, and p_C that:

Maximize $Expected\ Net\ Income = 5 + E(r_A)p_A + E(r_B)p_B + E(r_C)p_C$

Subject to: $TVAR_{1\%} = \frac{1}{1\%}\sum_S [5 + r_A p_A + r_B p_B + r_C p_C] p(r_A, r_B, r_C) \geq K$

where $S = \{(p_1, p_2, p_3, r_1, r_2, r_3, B)\ s.t.\ Net\ Income \geq VAR_{1\%}\}$

$p_A \geq 0$
$p_B \geq 0$
$p_C \geq 0$

$p_A + p_B + p_C \le 300$

Using equation 3.8, we write the Lagrange function as

$$\Lambda(p_A, p_B, p_C, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = p_A E(r_A) + p_B E(r_B) + p_C E(r_C) + 5$$
$$-\lambda_0 \left[\frac{1}{1\%} \sum_S [r_A p_A + r_B p_B + r_C p_C + 5] p(r_A, r_B, r_C) + L_0 \right]$$
$$-\lambda_1 p_A - \lambda_2 p_B - \lambda_3 p_C + \lambda_4 (p_A + p_B + p_C - 300)$$

The Adjusted RORAC Equilibrium Equation (3.14) for Problem 2 is written as:

$$\frac{p_A^*(E(r_A)+\lambda_4)}{\frac{1}{1\%}\sum_S r_A p_A p(r_A, r_B, r_C)} = \frac{p_B^*(E(r_B)+\lambda_4)}{\frac{1}{1\%}\sum_S r_B p_B p(r_A, r_B, r_C)} = \frac{p_C^*(E(r_C)+\lambda_4)}{\frac{1}{1\%}\sum_S r_C p_C p(r_A, r_B, r_C)}$$

The solutions to Problem 2 at various TVAR levels are shown in table 4.3.

<u>Problem 3 – Constraint: Premium for line C capped at 150M</u>

The problem is stated as follows:

Find p_A, p_B, and p_C that:

Maximize $\textit{Expected Net Income} = 5 + E(r_A) p_A + E(r_B) p_B + E(r_C) p_C$

Subject to: $TVAR_{1\%} = \frac{1}{1\%} \sum_S [5 + r_A p_A + r_B p_B + r_C p_C] p(r_A, r_B, r_C) \ge K$

where $S = \{(p_1, p_2, p_3, r_1, r_2, r_3, B)\ s.t.\ \textit{Net Income} \ge VAR_{1\%}\}$

$p_A \ge 0$

$p_B \ge 0$

$p_C \ge 0$

$p_C \le 150$

Using equation 3.8, we write the Lagrange function as

$$\Lambda(p_A, p_B, p_C, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = p_A E(r_A) + p_B E(r_B) + p_C E(r_C) + 5$$
$$-\lambda_0 \left[\frac{1}{1\%} \sum_S [r_A p_A + r_B p_B + r_C p_C + 5] p(r_A, r_B, r_C) + L_0 \right]$$
$$-\lambda_1 p_A - \lambda_2 p_B - \lambda_3 p_C + \lambda_4 (p_C - 150)$$

The Adjusted RORAC Equilibrium Equation (3.14) for Problem 3 is written as:

$$\frac{p_A^* E(r_A)}{\frac{1}{1\%}\sum_S r_A p_A p(r_A, r_B, r_C)} = \frac{p_B^* E(r_B)}{\frac{1}{1\%}\sum_S r_B p_B p(r_A, r_B, r_C)} = \frac{p_C^*(E(r_C)+\lambda_4)}{\frac{1}{1\%}\sum_S r_C p_C p(r_A, r_B, r_C)}$$

The solutions to Problem 3 at various TVAR levels are shown in table 4.4.

Table 4.1 - No Premium Constraint

Efficient Portfolios

							LOB Premium			TVAR Contribution			Net Income Contribution			RORAC		
Portfolio	TVAR	Net After Tax Income	RAROC	RORAC	Risk Leverage	Premium Leverage	Line A	Line B	Line C	Line A	Line B	Line C	Line A	Line B	Line C	Line A	Line B	Line C
1	25.0	17.9	7.2%	71.6%	0.10	0.29	26.5	23.1	22.9	2.5	5.2	22.4	1.1	2.2	9.6	43%	43%	43%
2	50.0	28.7	11.5%	57.3%	0.20	0.53	48.6	42.4	42.0	4.5	9.3	41.2	1.9	4.0	17.7	43%	43%	43%
3	75.0	39.4	15.8%	52.5%	0.30	0.77	70.7	61.7	61.1	6.6	13.6	59.9	2.8	5.9	25.7	43%	43%	43%
4	100.0	50.2	20.1%	50.2%	0.40	1.02	92.8	81.0	80.2	8.6	17.8	78.6	3.7	7.7	33.7	43%	43%	43%
5	125.0	60.9	24.4%	48.7%	0.50	1.26	114.9	100.3	99.3	10.7	22.4	96.9	4.6	9.5	41.8	43%	43%	43%
6	150.0	71.7	28.7%	47.8%	0.60	1.50	137.0	119.6	118.4	12.8	26.7	115.6	5.5	11.4	49.8	43%	43%	43%
7	175.0	82.4	33.0%	47.1%	0.70	1.74	159.1	138.9	137.4	14.8	30.5	134.7	6.4	13.2	57.8	43%	43%	43%
8	200.0	93.2	37.3%	46.6%	0.80	1.98	181.2	158.2	156.5	16.9	35.3	152.8	7.2	15.1	65.9	43%	43%	43%
9	225.0	103.9	41.6%	46.2%	0.90	2.23	203.3	177.5	175.6	18.9	39.0	172.1	8.1	16.9	73.9	43%	43%	43%
10	250.0	114.7	45.9%	45.9%	1.00	2.47	225.4	196.7	194.7	20.9	43.2	190.8	9.0	18.7	81.9	43%	43%	43%
11	275.0	125.4	50.2%	45.6%	1.10	2.71	247.5	216.0	213.8	23.0	47.5	209.5	9.9	20.6	90.0	43%	43%	43%
12	300.0	136.2	54.5%	45.4%	1.20	2.95	269.6	235.3	232.9	25.1	52.5	227.4	10.8	22.4	98.0	43%	43%	43%

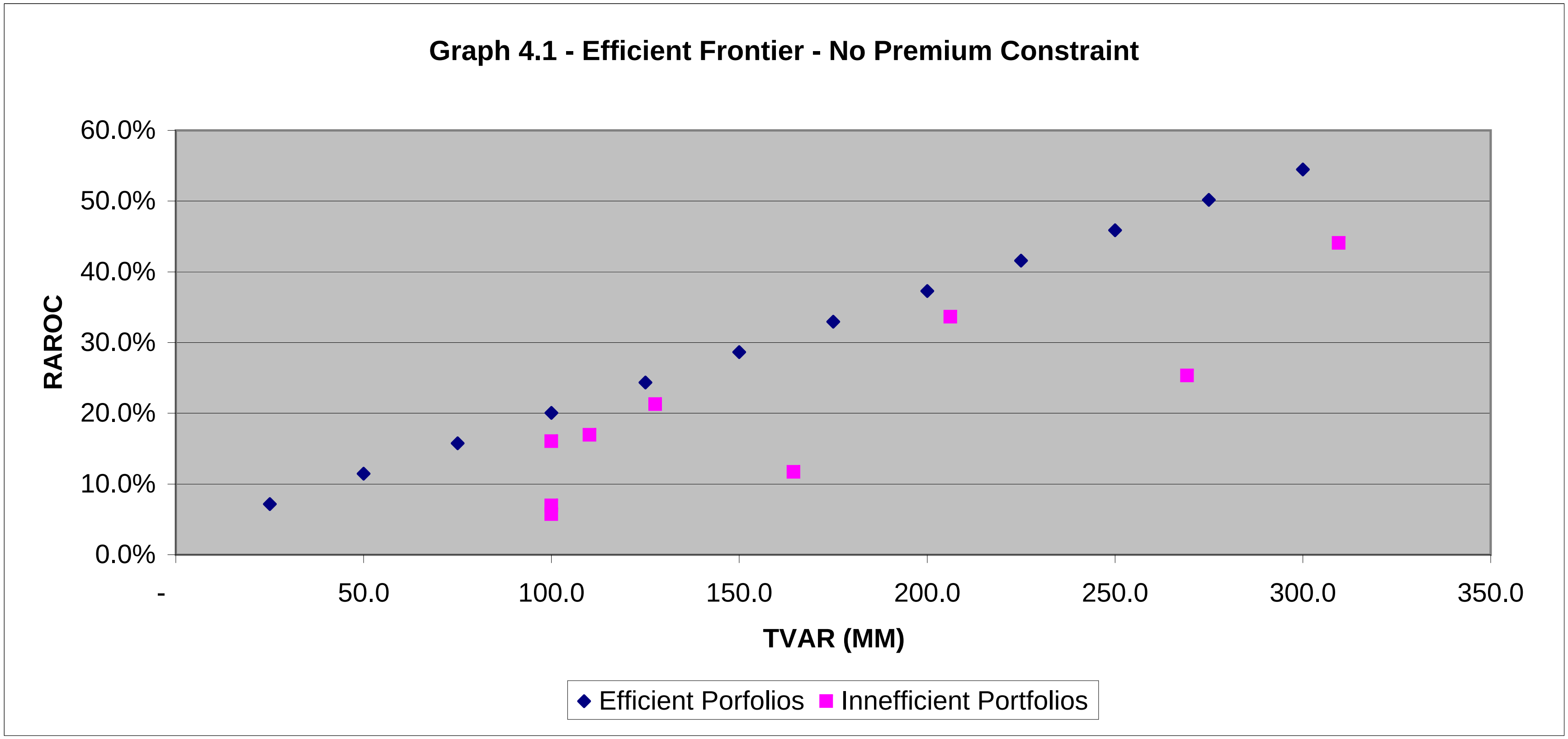
RAROC = Expected Net After Tax Income/250M
RORAC = Expected Net Income/TVAR
$\text{RORAC}_{\text{by Line}}$ = Expected Net Income Contribution/TVAR Contribution
Risk Leverage = TVAR/250M
Premium Leverage = Total Premium/250M
TVAR = ΣTVAR Contribution + 5M
Net After Tax Income = ΣIncome Contributions + 5M
* Negative of TVAR and TVAR Contribution are shown

Table 4.2 - No Premium Constraint

Inefficient Portfolios

							LOB Premium			TVAR Contribution			Net Income Contribution			RORAC		
Portfolio	TVAR	Net After Tax Income	RAROC	RORAC	Risk Leverage	Premium Leverage	Line A	Line B	Line C	Line A	Line B	Line C	Line A	Line B	Line C	Line A	Line B	Line C
13	100.0	40.1	16.0%	40.1%	0.40	0.33	-	-	83.4	-	-	105.0	-	-	35.1	N/A	N/A	33%
14	100.0	17.4	7.0%	17.4%	0.40	0.52	-	130.7	-	-	105.0	-	-	12.4	-	N/A	12%	N/A
15	100.0	14.3	5.7%	14.3%	0.40	0.93	233.5	-	-	105.0	-	-	9.3	-	-	9%	N/A	N/A
16	110.2	42.3	16.9%	38.4%	0.44	1.00	50.0	150.0	50.0	6.0	104.4	4.8	2.0	14.3	21.0	33%	14%	437%
17	127.6	53.2	21.3%	41.7%	0.51	1.00	-	175.0	75.0	-	103.3	29.4	-	16.7	31.6	N/A	16%	107%
18	309.6	110.2	44.1%	35.6%	1.24	1.00	-	-	250.0	-	-	314.6	-	-	105.2	N/A	N/A	33%
19	164.5	29.3	11.7%	17.8%	0.66	1.60	250.0	150.0	-	75.6	93.9	-	10.0	14.3	-	13%	15%	N/A
20	269.2	63.3	25.3%	23.5%	1.08	2.00	100.0	350.0	50.0	12.3	275.7	(13.9)	4.0	33.3	21.0	32%	12%	-152%
21	206.2	84.1	33.6%	40.8%	0.82	2.20	400.0	-	150.0	77.6	-	133.6	16.0	-	63.1	21%	N/A	47%

RAROC = Expected Net After Tax Income/250M
RORAC = Expected Net Income/TVAR
$\text{RORAC}_{\text{by Line}}$ = Expected Net Income Contribution/TVAR Contribution
Risk Leverage = TVAR/250M
Premium Leverage = Total Premium/250M
TVAR = ΣTVAR Contribution + 5M
Net After Tax Income = ΣIncome Contributions + 5M
* Negative of TVAR and TVAR Contribution are shown



* Negative of TVAR and TVAR Contribution are shown

Table 4.3 - Total Premium for lines A, B, and C capped at 300M

Efficient Portfolios

Portfolio	TVAR	Net After Tax Income	RAROC	RORAC	Risk Leverage	Premium Leverage	Premium Contribution Line A	Premium Contribution Line B	Premium Contribution Line C	TVAR Contribution Line A	TVAR Contribution Line B	TVAR Contribution Line C
1	25.0	17.9	7.2%	71.6%	0.10	0.29	26.5	23.1	22.9	2.5	5.2	22.4
2	50.0	28.7	11.5%	57.3%	0.20	0.53	48.6	42.4	42.0	4.5	9.4	41.1
3	75.0	39.4	15.8%	52.5%	0.30	0.77	70.7	61.7	61.1	6.6	13.8	59.6
4	100.0	50.2	20.1%	50.2%	0.40	1.02	92.8	81.0	80.2	8.7	18.1	78.3
22	125.0	60.8	24.3%	48.7%	0.50	1.20	96.2	103.6	100.1	7.6	22.2	100.3
23	150.0	70.6	28.2%	47.1%	0.60	1.20	47.7	130.4	121.8	2.1	24.7	128.3
24	175.0	79.8	31.9%	45.6%	0.70	1.20	2.1	155.5	142.5	0.0	26.6	153.3
25	200.0	87.8	35.1%	43.9%	0.80	1.20	-	133.4	166.6	-	8.0	197.0
26	225.0	94.7	37.9%	42.1%	0.90	1.20	-	112.1	187.9	-	0.6	229.4
27	250.0	101.2	40.5%	40.5%	1.00	1.20	-	92.1	207.9	-	(2.9)	257.9
28	275.0	107.6	43.0%	39.1%	1.10	1.20	-	72.7	227.3	-	(4.1)	284.1
29	300.0	113.7	45.5%	37.9%	1.20	1.20	-	53.8	246.2	-	(4.0)	309.0

Portfolio	Net Income Contribution Line A	Net Income Contribution Line B	Net Income Contribution Line C	RORAC Line A	RORAC Line B	RORAC Line C	λ_4	Adjusted RORAC Line A	Adjusted RORAC Line B	Adjusted RORAC Line C
1	1.1	2.2	9.6	43%	43%	43%	0%	43%	43%	43%
2	1.9	4.0	17.7	43%	43%	43%	0%	43%	43%	43%
3	2.8	5.9	25.7	43%	43%	43%	0%	43%	43%	43%
4	3.7	7.7	33.7	43%	43%	43%	0%	43%	43%	43%
22	3.8	9.9	42.1	51%	45%	42%	-1%	41%	41%	41%
23	1.9	12.4	51.3	93%	50%	40%	-2%	38%	38%	38%
24	0.1	14.8	59.9	187%	56%	39%	-3%	37%	37%	36%
25	-	12.7	70.1	N/A	159%	36%	-8%	N/A	29%	29%
26	-	10.7	79.0	N/A	1903%	34%	-9%	N/A	27%	27%
27	-	8.8	87.5	N/A	-299%	34%	-10%	N/A	26%	26%
28	-	6.9	95.6	N/A	-169%	34%	-11%	N/A	25%	25%
29	-	5.1	103.6	N/A	-129%	34%	-11%	N/A	24%	25%

RAROC = Expected Net After Tax Income/250M
RORAC = Expected Net Income/TVAR
$\text{RORAC}_{\text{by Line}}$ = Expected Net Income Contribution/TVAR Contribution
Adjusted RORAC = (Net Income Contribution - λ_4 X Premium Contribution)/TVAR Contribution
Risk Leverage = TVAR/250M
Premium Leverage = Total Premium/250M
TVAR = ΣTVAR Contribution + 5M
Net After Tax Income = ΣIncome Contributions + 5M
* Negative of TVAR and TVAR Contribution are shown

Table 4.4 - Total Premium for line C capped at 150M

Efficient Portfolios

Portfolio	TVAR	Net After Tax Income	RAROC	RORAC	Risk Leverage	Premium Leverage	Premium Contribution			TVAR Contribution			Net Income Contribution			RORAC			λ_4	Adjusted RORAC		
							Line A	Line B	Line C	Line A	Line B	Line C	Line A	Line B	Line C	Line A	Line B	Line C		Line A	Line B	Line C
1	25.0	17.9	7.2%	71.6%	0.10	0.29	26.5	23.1	22.9	2.5	5.2	22.4	1.1	2.2	9.6	43%	43%	43%	0%	43%	43%	43%
2	50.0	28.7	11.5%	57.3%	0.20	0.53	48.6	42.4	42.0	4.5	9.4	41.1	1.9	4.0	17.7	43%	43%	43%	0%	43%	43%	43%
3	75.0	39.4	15.8%	52.5%	0.30	0.77	70.7	61.7	61.1	6.6	13.8	59.6	2.8	5.9	25.7	43%	43%	43%	0%	43%	43%	43%
4	100.0	50.2	20.1%	50.2%	0.40	1.02	92.8	81.0	80.2	8.7	18.1	78.3	3.7	7.7	33.7	43%	43%	43%	0%	43%	43%	43%
5	125.0	60.9	24.4%	48.7%	0.50	1.26	114.9	100.3	99.3	10.7	22.4	96.9	4.6	9.5	41.8	43%	43%	43%	0%	43%	43%	43%
6	150.0	71.7	28.7%	47.8%	0.60	1.50	137.0	119.6	118.4	12.7	26.3	116.0	5.5	11.4	49.8	43%	43%	43%	0%	43%	43%	43%
7	175.0	82.4	33.0%	47.1%	0.70	1.74	159.1	138.9	137.4	14.8	30.5	134.7	6.4	13.2	57.8	43%	43%	43%	0%	43%	43%	43%
30	200.0	92.7	37.1%	46.4%	0.80	2.11	205.0	172.7	150.0	23.7	47.0	134.3	8.2	16.4	63.1	35%	35%	47%	-11%	35%	35%	35%
31	225.0	99.8	39.9%	44.4%	0.90	2.58	277.8	216.7	150.0	45.9	85.3	98.8	11.1	20.6	63.1	24%	24%	64%	-26%	24%	24%	24%
32	250.0	105.4	42.2%	42.2%	1.00	2.94	331.8	252.4	150.0	64.7	116.8	73.5	13.3	24.0	63.1	20%	21%	86%	-32%	20%	21%	21%
33	275.0	110.3	44.1%	40.1%	1.10	3.25	377.7	284.1	150.0	81.3	146.1	52.6	15.1	27.0	63.1	19%	19%	120%	-36%	19%	19%	19%
34	300.0	114.8	45.9%	38.3%	1.20	3.55	425.3	311.5	150.0	96	168.1	40.7	17	29.7	63.1	18%	18%	155%	-37%	18%	18%	18%

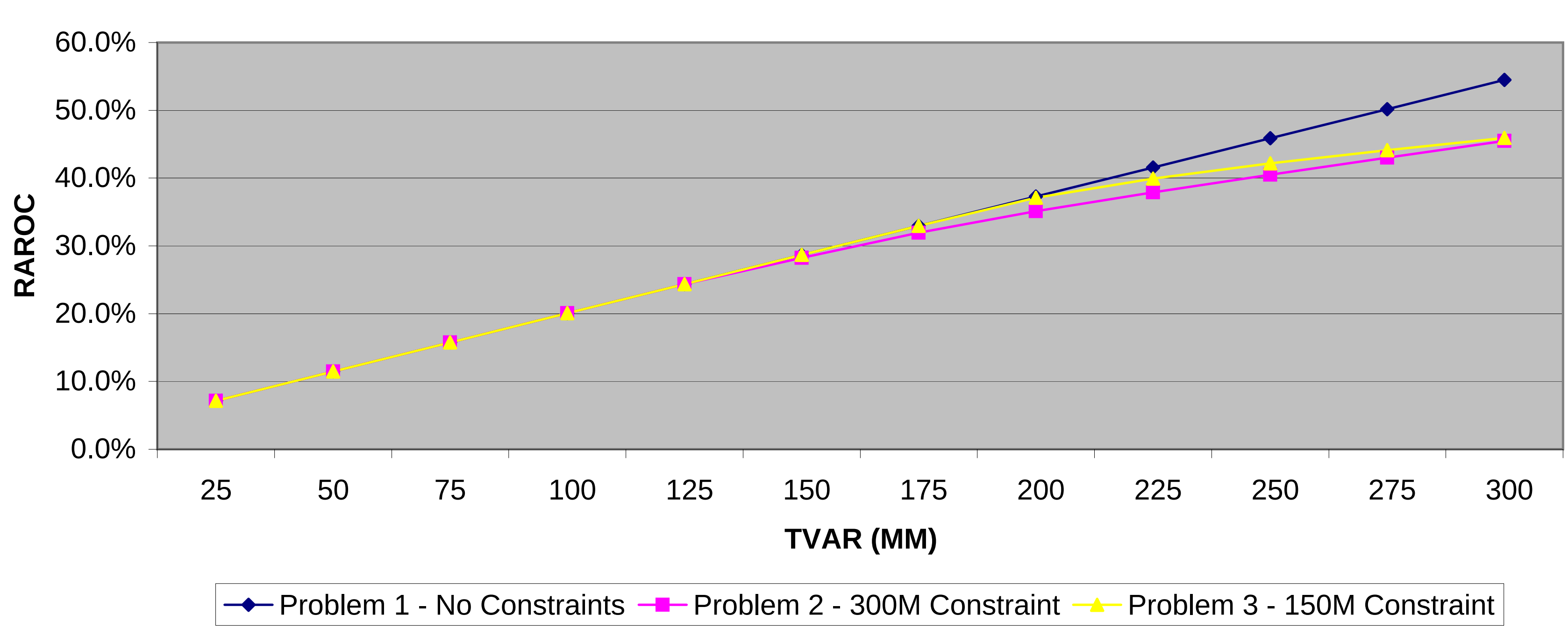
RAROC = Expected Net After Tax Income/250M
RORAC = Expected Net Income/TVAR
RORAC $_{\text{by Line}}$ = Expected Net Income Contribution/TVAR Contribution
Adjusted RORAC = (Net Income Contribution - λ_4 X Premium Contribution)/TVAR Contribution
Risk Leverage = TVAR/250M
Premium Leverage = Total Premium/250M
TVAR = ΣTVAR Contribution + 5M
Net After Tax Income = ΣIncome Contributions + 5M
* Negative of TVAR and TVAR Contribution are shown

Table 4.5 - Efficient Frontier Comparison

	Expected Net Income			RAROC = Expected Net Income / 250M		
TVAR	Problem 1 - No Constraints	Problem 2 - 300M Constraint	Problem 3 - 150M Constraint	Problem 1 - No Constraints	Problem 2 - 300M Constraint	Problem 3 - 150M Constraint
25	17.9	17.9	17.9	7.2%	7.2%	7.2%
50	28.7	28.7	28.7	11.5%	11.5%	11.5%
75	39.4	39.4	39.4	15.8%	15.8%	15.8%
100	50.2	50.2	50.2	20.1%	20.1%	20.1%
125	60.9	60.8	60.9	24.4%	24.3%	24.4%
150	71.7	70.6	71.7	28.7%	28.2%	28.7%
175	82.4	79.8	82.4	33.0%	31.9%	33.0%
200	93.2	87.8	92.7	37.3%	35.1%	37.1%
225	103.9	94.7	99.8	41.6%	37.9%	39.9%
250	114.7	101.2	105.4	45.9%	40.5%	42.2%
275	125.4	107.6	110.3	50.2%	43.0%	44.1%
300	136.2	113.7	114.8	54.5%	45.5%	45.9%

* Negative of TVAR and TVAR Contribution are shown

Graph 4.2 - Efficient Frontier Comparison



* Negative of TVAR and TVAR Contribution are shown

Observations

The preceding tables and graphs lead to several interesting observations. Here we highlight a few:

- At the portfolio level, RORAC, as we have defined it, is an incomplete measure of profitability. The risk leverage tells the rest of the story. Table 4.1 shows that RORAC and RAROC have in fact an inverse relationship on the efficient frontier. Portfolio 1's RORAC of 71.6% would not necessarily endear it to shareholders willing to accept more risk as they are realizing only a 7.2% return on their investment.
- At the line of business level, RORAC, as we have defined it, is a poor and perhaps misleading measure of a line's profitability or even relative profitability. In table 4.3, for instance, Portfolio 23 produces RORAC of 93%, 50%, and 40% for lines A, B, and C, respectively. As it turns out, each line is performing exactly as it should given the constraints under which the company is operating, and the portfolio is optimal. The Adjusted RORACs are in perfect equilibrium. Is line A the most profitable one of the three? Should the company binge on line A? The answer to both questions is obviously no. In table 4.4, the RORAC for line C increases from 47% to 155% from Portfolio 30 through 34. Yet, the premium for line C remains unchanged for these five portfolios, which are all optimal. Is line C suddenly more profitable in portfolio 34 than it was in the previous 4? Again, the answer is no.
- The preceding tables further illustrate why Premium Leverage is a poor and perhaps misleading measure of portfolio risk. As the premium leverage does not take into account the unique characteristics of a portfolio, using premium leverage even as a rough gauge of risk is inappropriate. For instance, in table 4.2, we show several portfolios with a premium leverage of 1.00, yet with drastically different risk leverage. Worse, comparing portfolio 6 in table 4.1 to portfolio 18 in table 4.2 shows that while Portfolio 6 has 50% more premium leverage, it has about half the risk of portfolio 21.

Conclusion

This paper does not present a new approach to dealing with the Capital Allocation problem as much as it frames the problem in a different light. Many of the elegant and clever Capital Allocation schemes that have been published, including Myers and Read's [5] or, more recently, Bodoff's [1], are in fact Risk Capital decomposition schemes. It has been argued that many of these schemes lack an economic foundation and are rather arbitrary. Worse, these schemes could lead to the wrong decisions. As we have observed above, RORAC measures can be misleading indicators of profitability.

Our approach is based on one fundamental economic principle: **Companies aim to maximize shareholder wealth**. In doing so, they have to navigate various constraints that are imposed on them. When the TVAR is used as a risk constraint, it leads to Equilibrium Equations (2.11) and (3.14). These equations are a byproduct of our approach but not its foundation. Similar results may be obtained with other conditional expected value measures. In fact, the optimization approach is still valid even when equations (2.11) and (3.14) do not hold as would be the case, say, with a Value at Risk measure.

We have left many questions unanswered in this paper. Most notably, we have not broached the question of whether the solutions to the optimization problem are unique nor have we investigated what happens when the competitive market assumption is violated. Nevertheless, we hope this framing of the Capital Allocation question will spur further debate and that new ideas will emerge in the process.

APPENDIX

Interpretation of the Lagrange Multipliers

We introduce functions $h(p_1, p_2)$ and $g_i(p_1, p_2) \quad i = 0,1,\cdots,5$, and rewrite equations (3.1’) through (3.7’) and (3.8) as follows:

Maximize: $h(p_1, p_2) = p_1E(r_1) + p_2E(r_2) + E(B)$ A1

subject to: $g_0(p_1, p_2) = -\frac{1}{\beta}\iint_C (p_1r_1 + p_2r_2 + B) f(r_1, r_2) d_{r_1} d_{r_2} \le -L_0$ A2

$g_1(p_1, p_2) = -p_1 \le 0$ A3

$g_2(p_1, p_2) = -p_2 \le 0$ A4

$g_3(p_1, p_2) = p_1 \le L_3$ A5

$g_4(p_1, p_2) = p_2 \le L_4$ A6

$g_5(p_1, p_2) = p_1 + p_2 \le L_5$ A7

$$\Lambda(p_1, p_2, \lambda_0, \lambda_1, \lambda_2) = h(p_1, p_2) + \lambda_0 g_0(p_1, p_2) + \lambda_1 g_1(p_1, p_2) + \lambda_2 g_2(p_1, p_2) + \lambda_3 g_3(p_1, p_2) + \lambda_4 g_4(p_1, p_2) + \lambda_5 g_5(p_1, p_2) \quad \text{A8}$$

Since we assume that $\lambda_1 = \lambda_2 = 0$, we rewrite (A8) as

$$\Lambda(p_1, p_2, \lambda_0, \lambda_1, \lambda_2) = h(p_1, p_2) + \lambda_0 g_0(p_1, p_2) + \lambda_3 g_3(p_1, p_2) + \lambda_4 g_4(p_1, p_2) + \lambda_5 g_5(p_1, p_2) \quad \text{A9}$$

For simplicity, we drop the arguments in the functions and write (A9) as:

$$\Lambda = h + \lambda_0 g_0 + \lambda_3 g_3 + \lambda_4 g_4 + \lambda_5 g_5 \quad \text{A10}$$

Derivation of λ_0^*

Using the chain rule, we take the partial derivative of h with respect to L_0 and obtain:

$$\frac{\partial h}{\partial L_0} = \frac{\partial h}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial h}{\partial p_2}\frac{\partial p_2}{\partial L_0} \quad \text{A11}$$

Using the chain rule and equation (A2) we obtain:

$$\frac{\partial g_0}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial g_0}{\partial p_2}\frac{\partial p_2}{\partial L_0} + 1 = 0 \quad \text{A12}$$

Multiplying (A12) above by λ_0, we get:

$$\lambda_0 \frac{\partial g_0}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_0 \frac{\partial g_0}{\partial p_2}\frac{\partial p_2}{\partial L_0} + \lambda_0 = 0 \quad \text{A13}$$

Similarly, we obtain the following from A5, A6, and A7:

$$\frac{\partial g_3}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial g_3}{\partial p_2}\frac{\partial p_2}{\partial L_0} = 0 \quad \text{A14}$$

$$\lambda_3 \frac{\partial g_3}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_3 \frac{\partial g_3}{\partial p_2}\frac{\partial p_2}{\partial L_0} = 0 \qquad \text{A15}$$

$$\frac{\partial g_4}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial g_4}{\partial p_2}\frac{\partial p_2}{\partial L_0} = 0 \qquad \text{A16}$$

$$\lambda_4 \frac{\partial g_4}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_4 \frac{\partial g_4}{\partial p_2}\frac{\partial p_2}{\partial L_0} = 0 \qquad \text{A17}$$

$$\frac{\partial g_5}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial g_5}{\partial p_2}\frac{\partial p_2}{\partial L_0} = 0 \qquad \text{A18}$$

$$\lambda_5 \frac{\partial g_5}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_5 \frac{\partial g_5}{\partial p_2}\frac{\partial p_2}{\partial L_0} = 0 \qquad \text{A19}$$

Adding (A13), A(15), A(17), and A(19) to A(11) yields:

$$\begin{aligned}\frac{\partial h}{\partial L_0} &= \frac{\partial h}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial h}{\partial p_2}\frac{\partial p_2}{\partial L_0} + \lambda_0 \frac{\partial g_0}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_0 \frac{\partial g_0}{\partial p_2}\frac{\partial p_2}{\partial L_0} + \lambda_0 + \lambda_3 \frac{\partial g_3}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_3 \frac{\partial g_3}{\partial p_2}\frac{\partial p_2}{\partial L_0} \\ &\quad + \lambda_4 \frac{\partial g_4}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_4 \frac{\partial g_4}{\partial p_2}\frac{\partial p_2}{\partial L_0} + \lambda_5 \frac{\partial g_5}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \lambda_5 \frac{\partial g_5}{\partial p_2}\frac{\partial p_2}{\partial L_0} \qquad \text{A20}\end{aligned}$$

or

$$\begin{aligned}\frac{\partial h}{\partial L_0} &= \left(\frac{\partial h}{\partial p_1} + \lambda_0 \frac{\partial g_0}{\partial p_1} + \lambda_3 \frac{\partial g_3}{\partial p_1} + \lambda_4 \frac{\partial g_4}{\partial p_1} + \lambda_5 \frac{\partial g_5}{\partial p_1}\right)\frac{\partial p_1}{\partial L_0} \\ &\quad + \left(\frac{\partial h}{\partial p_2} + \lambda_0 \frac{\partial g_0}{\partial p_2} + \lambda_3 \frac{\partial g_3}{\partial p_2} + \lambda_4 \frac{\partial g_4}{\partial p_2} + \lambda_5 \frac{\partial g_5}{\partial p_2}\right)\frac{\partial p_2}{\partial L_0} + \lambda_0\end{aligned}$$

or

$$\frac{\partial h}{\partial L_0} = \frac{\partial}{\partial p_1}\left(h + \lambda_0 g_0 + \lambda_3 g_3 + \lambda_4 g_4 + \lambda_5 g_5\right)\frac{\partial p_1}{\partial L_0} + \frac{\partial}{\partial p_2}\left(h + \lambda_0 g_0 + \lambda_3 g_3 + \lambda_4 g_4 + \lambda_5 g_5\right)\frac{\partial p_2}{\partial L_0} + \lambda_0$$

or

$$\frac{\partial h}{\partial L_0} = \frac{\partial(\Lambda)}{\partial p_1}\frac{\partial p_1}{\partial L_0} + \frac{\partial(\Lambda)}{\partial p_2}\frac{\partial p_2}{\partial L_0} + \lambda_0$$

Since $\frac{\partial(\Lambda)}{\partial p_1} = 0$ and $\frac{\partial(\Lambda)}{\partial p_2} = 0$ at the optimal point, we have

$$\frac{\partial h}{\partial L_0} = \lambda_0^* \qquad \text{A21}$$

Derivation of λ_3^*

Using the chain rule, we take the partial derivative of h with respect to L_3 and obtain:

$$\frac{\partial h}{\partial L_3} = \frac{\partial h}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \frac{\partial h}{\partial p_2}\frac{\partial p_2}{\partial L_3} \qquad \text{A22}$$

Using the chain rule and equation (A5) we obtain:

$$\frac{\partial g_3}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \frac{\partial g_3}{\partial p_2}\frac{\partial p_2}{\partial L_3} - 1 = 0; \qquad \text{A23}$$

Multiplying (A23) above by λ_3, we get:

$$\lambda_3 \frac{\partial g_3}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_3 \frac{\partial g_3}{\partial p_2}\frac{\partial p_2}{\partial L_3} - \lambda_3 = 0 \qquad \text{A24}$$

Similarly, we obtain the following from A2, A6, and A7:

$$\frac{\partial g_0}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \frac{\partial g_0}{\partial p_2}\frac{\partial p_2}{\partial L_3} = 0 \qquad \text{A25}$$

$$\lambda_0 \frac{\partial g_0}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_0 \frac{\partial g_0}{\partial p_2}\frac{\partial p_2}{\partial L_3} = 0 \qquad \text{A26}$$

$$\frac{\partial g_4}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \frac{\partial g_4}{\partial p_2}\frac{\partial p_2}{\partial L_3} = 0 \qquad \text{A27}$$

$$\lambda_4 \frac{\partial g_4}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_4 \frac{\partial g_4}{\partial p_2}\frac{\partial p_2}{\partial L_3} = 0 \qquad \text{A28}$$

$$\frac{\partial g_5}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \frac{\partial g_5}{\partial p_2}\frac{\partial p_2}{\partial L_3} = 0 \qquad \text{A29}$$

$$\lambda_5 \frac{\partial g_5}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_5 \frac{\partial g_5}{\partial p_2}\frac{\partial p_2}{\partial L_3} = 0 \qquad \text{A30}$$

Adding (A24), A(26), A(28), and A(30) to A(22) yields:

$$\begin{aligned}\frac{\partial h}{\partial L_3} = {} & \frac{\partial h}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \frac{\partial h}{\partial p_2}\frac{\partial p_2}{\partial L_3} + \lambda_0 \frac{\partial g_0}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_0 \frac{\partial g_0}{\partial p_2}\frac{\partial p_2}{\partial L_3} + \lambda_3 \frac{\partial g_3}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_3 \frac{\partial g_3}{\partial p_2}\frac{\partial p_2}{\partial L_3} - \lambda_3 \\ & + \lambda_4 \frac{\partial g_4}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_4 \frac{\partial g_4}{\partial p_2}\frac{\partial p_2}{\partial L_3} + \lambda_5 \frac{\partial g_5}{\partial p_1}\frac{\partial p_1}{\partial L_3} + \lambda_5 \frac{\partial g_5}{\partial p_2}\frac{\partial p_2}{\partial L_3} \qquad \text{A31}\end{aligned}$$

or

$$\frac{\partial h}{\partial L_3}=\left(\frac{\partial h}{\partial p_1}+\lambda_0\frac{\partial g_0}{\partial p_1}+\lambda_3\frac{\partial g_3}{\partial p_1}+\lambda_4\frac{\partial g_4}{\partial p_1}+\lambda_5\frac{\partial g_5}{\partial p_1}\right)\frac{\partial p_1}{\partial L_3}$$
$$+\left(\frac{\partial h}{\partial p_2}+\lambda_0\frac{\partial g_0}{\partial p_2}+\lambda_3\frac{\partial g_3}{\partial p_2}+\lambda_4\frac{\partial g_4}{\partial p_2}+\lambda_5\frac{\partial g_5}{\partial p_2}\right)\frac{\partial p_2}{\partial L_3}-\lambda_3$$

or

$$\frac{\partial h}{\partial L_3}=\frac{\partial}{\partial p_1}\left(h+\lambda_0 g_0+\lambda_3 g_3+\lambda_4 g_4+\lambda_5 g_5\right)\frac{\partial p_1}{\partial L_3}+\frac{\partial}{\partial p_2}\left(h+\lambda_0 g_0+\lambda_3 g_3+\lambda_4 g_4+\lambda_5 g_5\right)\frac{\partial p_2}{\partial L_3}-\lambda_3$$

or

$$\frac{\partial h}{\partial L_3}=\frac{\partial(\Lambda)}{\partial p_1}\frac{\partial p_1}{\partial L_3}+\frac{\partial(\Lambda)}{\partial p_2}\frac{\partial p_2}{\partial L_3}-\lambda_3$$

Since $\frac{\partial(\Lambda)}{\partial p_1}=0$ and $\frac{\partial(\Lambda)}{\partial p_2}=0$ at the optimal point, we have

$$\frac{\partial h}{\partial L_3}=-\lambda_3^* \qquad \text{A32}$$

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