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Stochastic Ordering of Reinsurance Structures

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Abstract

The paper offers a simple framework for ranking the common reinsurance structures in practice with the theory of stochastic orders. The basic idea is to slice the space of reinsurance structures into groups by expected loss cost to facilitate the comparisons within the group and between groups. Given the standard risk aversion assumption in economics, a spectrum of reinsurance structures with the same expected loss cost can be compared analytically with one another and sequenced based on their risk coverages under the convex order. The paper then expands the dimension of the comparison to groups of reinsurance structures with different expected loss costs, which can be ranked under the increasing convex order and the usual stochastic order. As such, the paper maps out the ordering for the entire space of reinsurance structures and presents it in a matrix format for quick reference. The implication of this stochastic ordering to reinsurance pricing is also investigated.

Keywords: Reinsurance; Usual Stochastic Order; Convex Order; Increasing Convex Order; Stochastic Dominance; Insurance Premium Principles.

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1 Introduction

Reinsurance is one of the most frequently used risk management tools by insurance companies in managing their portfolios. Insurance companies regularly evaluate and, if necessary, modify the structure of their reinsurance program to adjust their overall risk exposures in an evolving business environment. For example, an enterprise risk management (ERM) analysis may compare the coverages and the efficiency between the current reinsurance program and alternative reinsurance structures. These alternative reinsurance structures may involve increasing or decreasing the retention level of an excess of loss reinsurance, adding an aggregate deductible or an aggregate limit and adjusting the placement ratio.

To find the optimal reinsurance *contract* that maximizes an objective variable, such as the net underwriting income, the typical industry approach is to run a simulation model with as many potential reinsurance structures as possible. One of the key challenges in the ERM evaluation process is how to set the reinsurance prices for these alternative options, which to a large extent determines the efficiencies of the options. Given that the ERM modelers usually do not have the benefit of market quotes for all the options, it is important these reinsurance structures can be properly ordered and priced in the model. The abundance of reinsurance choices together with the complexity of reinsurance pricing, however, often makes the selection process very difficult.

The goal of this paper is to provide actuaries, underwriters and brokers a framework to compare common reinsurance structures so that unnecessary simulation may be avoided and reasonable results can be obtained quickly in an ERM analysis. We first explore the risk ranking of common reinsurance structures using the convex order from the theory of stochastic orders (e.g., Shaked and Shanthikumar 2007, Müller and Stoyan 2002 and Denuit et al. 2005). We then further expand the dimension of the comparison to reinsurance structures with different expected loss costs using the usual stochastic order (equivalently, the first-order stochastic dominance)[1] and the increasing convex order (dual to the general second-order stochastic dominance).

The convex order is dual to the concave order, which is the familiar Rothschild-Stiglitz second-order stochastic dominance (R-S SSD) with equal means as pioneered by Rothschild and Stiglitz (1970) in economics. Heyer (2001) uses the general SSD to rank reinsurance *contracts* on an empirical distribution basis through simulation. Assuming a risk-averse principal (or equivalently an increasing concave utility function), if the net underwriting income resulting from reinsurance structure A is larger in "size" and less volatile than reinsurance structure B, then A is second-order stochastic dominating B from a cedant's point of view. However, the result of the underwriting income comparison using the general SSD is often inconclusive as demonstrated in Heyer's analysis. This paper will focus on the loss distributions, rather than the underwriting income distributions, of the reinsurance structures as there exists a natural ordering for the former but not necessarily for the latter.

[1] See Levy (1998) for a general introduction to stochastic dominance and see Heyer (2001) for an application of stochastic dominance to reinsurance.

The convex order allows us to compare alternatives that have the same expected value and thus eliminate the need to compare "size" or "magnitude." The focus of the comparison, instead, can then be on the "variability" or the pure risk of the reinsurance structures. We will show analytically that any risk-averse individual under the convex order can distinguish and rank basic reinsurance structures given their natural orders in "variability." In short, under the convex order, the stop-loss reinsurance is more risky than the quota share reinsurance, which in turn is more risky than the reinsurance with an aggregate limit (i.e., 100 percent quota share with a cap):

$$\textbf{Aggregate Limit} \preceq_{cx} \textbf{Quota Share} \preceq_{cx} \textbf{Stop-Loss}$$

where $A \preceq_{cx} B$ means B dominates A under the convex order.

This line of reasoning can be extended to analyzing the aggregate loss treaties with more than one contract feature. For example, a quota share treaty with a stop-loss threshold can be compared with a quota share treaty with an aggregate limit. More parameters need to be calibrated within a treaty to make sure that the mean loss is the same across all treaties as required by the convex order. Note that these combination structures with two contract features form a continuum of options that are bounded by the three basic reinsurance structures. Outlined below are the rankings of some possible combinations.

$$\begin{aligned}
&\textbf{Aggregate Limit}\\
\preceq_{cx}&\textbf{Mixture of Quota Share and Aggregate Limit}\\
\preceq_{cx}&\textbf{Quota Share}\\
\preceq_{cx}&\textbf{Mixture of Stop-Loss and Quota Share}\\
\preceq_{cx}&\textbf{Stop-Loss.}
\end{aligned}$$

The approaches we have used in analyzing the aggregate loss reinsurance can also be applied to the excess of loss (XOL) reinsurance treaties with features such as annual aggregate deductible (AAD), higher per claim retention[2], partial placement (or equivalently cedant co-participation) and aggregate limit. Note that the convex order is closed under convolutions. That is, when the claim count distribution is independent of the severity distributions, the dominance relationship between the severity distributions at the per risk/per occurrence level can be carried over to the aggregate layer loss level. This closure property is crucial in proving the relationship between XOL with partial placement and XOL with higher retention.

We will show that under the convex order, these XOL reinsurance treaties along

[2]Here the per risk/per occurrence retention is raised, but the sum of the retention and limit is the same as that for the original layer. See Definition 5.6.

with the corresponding hybrid structures can be ranked analytically as follows:

$$
\begin{aligned}
&\textbf{XOL with Aggregate Limit}\\
\preceq_{cx}&\textbf{XOL with Mixture of Partial Placement and Aggregate Limit}\\
\preceq_{cx}&\textbf{XOL with Partial Placement}\\
\preceq_{cx}&\textbf{XOL with Mixture of Higher Retention and Partial Placement}\\
\preceq_{cx}&\textbf{XOL with Higher Retention}\\
\preceq_{cx}&\textbf{XOL with Mixture of Aggregate Deductible and Higher Retention}\\
\preceq_{cx}&\textbf{XOL with Aggregate Deductible.}
\end{aligned}
$$

The next step is to expand the dimension of the comparison to reinsurance structures with different expected loss costs using the usual stochastic order (equivalently, the first-order stochastic dominance) and the increasing convex order (the dual to the general second-order stochastic dominance). The usual stochastic order ($\preceq_{st}$) can be established between any two structures that are of the same type, but have different expected losses. If different types of structures are involved in the one-on-one comparison, we may be able to establish dominance under the weaker increasing convex order ($\preceq_{icx}$).

The use of the usual stochastic order and increasing convex order greatly expands the range of reinsurance structures that can be compared and ranked. While it appears that the number of comparison combinations may be infinite, some reinsurance treaties, however, are not comparable under any of the three stochastic orders. Particularly, the comparison is inconclusive between a quota share treaty and a treaty with both an aggregate limit and an aggregate deductible. The reason for inconclusiveness is that neither treaty has thicker tails on both ends of the density function, which is required for the dominance relationship. We will show that the inconclusiveness follows a predictable pattern based on the types of reinsurance structure.

Section 2 of the paper defines the three stochastic orders and Section 3 compares the risk rankings of basic reinsurance structures under the convex order. The paper then extends the analysis to the reinsurance structures with different expected values in Section 4 while Section 5 applies the same methodology to excess of loss reinsurance. We then compare aggregate reinsurance structures with XOL reinsurance structures in Section 6. The implications of this risk-ranking analysis to reinsurance pricing and the optimal reinsurance literature are considered in Section 7 and the concluding remarks are in Section 8.

2 Preliminaries

Assume a standard collective risk model where $x > 0$ is a continuous ground-up loss random variable for a single occurrence or a single risk with mean $0 < E(x) < \infty$ and variance $0 < Var(x) < \infty$. Let $N \geq 0$ be an integer-based random variable for the ground-up loss frequency and independent of x. S denotes the corresponding aggregate loss and $S = \sum_{i=1}^{N} x_i$, where i is the index for N and $S = 0$ when $N = 0$.

We first define the usual stochastic order and then introduce the increasing convex order and the convex order.

Definition 2.1. Usual Stochastic Order: *(Shaked and Shanthikumar 2007, Definition 1.A.1)* Let X and Y be two random variables such that $P(X > t) \leq P(Y > t)$ for all $t \in (-\infty, \infty)$. Then X is said to be smaller than Y in the usual stochastic order or $X \preceq_{st} Y$.

In economics, the usual stochastic order is called the first-order stochastic dominance (FSD). The definition implies that at every percentile, Y has a higher value than X. It can be characterized as $X \preceq_{st} Y$ if, and only if, $E(\phi(X)) \leq E(\phi(Y))$ for all non-decreasing functions $\phi : R \to R$, provided the expectations exist. Clearly, if $X \preceq_{st} Y$, then $E(X) \leq E(Y)$ and $Var(X) \leq Var(Y)$ as both the expectation and the variance functions are non-decreasing.

Definition 2.2. Increasing Convex Order: *(Shaked and Shanthikumar 2007, Definition 4.A.1)* Let X and Y be two random variables such that $E(\phi(X)) \leq E(\phi(Y))$ for all increasing convex functions $\phi : R \to R$, provided the expectations exist. Then X is said to be smaller than Y in the increasing convex order or $X \preceq_{icx} Y$.

The increasing convex order is a dual order to the increasing concave order or the second-order stochastic dominance (Kass et al. 2009, Theorem 7.3.10), which is often used by financial economists to analyze investment decision-making under uncertainty. In other words, if a risk-averse individual prefers Y to X under the second-order stochastic dominance, he/she would equivalently also prefer $-X$ to $-Y$ under the increasing convex order. Thus it is usually a matter of convenience and intuition to use the increasing convex order rather than the increasing concave order or the second-order stochastic dominance when the objects for comparison are losses rather than assets.

Definition 2.3. Convex Order: *(Shaked and Shanthikumar 2007, Definition 3.A.1)* Let X and Y be two random variables such that $E(\phi(X)) \leq E(\phi(Y))$ for all convex functions $\phi : R \to R$, provided the expectations exist. Then X is said to be smaller than Y in the convex order or $X \preceq_{cx} Y$.

The convex order is closely related to the increasing convex order and second-order stochastic dominance. The difference between the convex order and the increasing convex order is that the convex order requires that $E(\phi(X)) \leq E(\phi(Y))$ holds for *all* convex functions ϕ. Since $\phi(x) = x$ and $\phi(x) = -x$ are both convex, $X \preceq_{cx} Y$ implies that X and Y must have the same expected value, i.e., $E(X) = E(Y)$.

In a sense, the increasing convex order compares both the "size" and the "variability" of random variables while the convex order compares only the "variability," given that the underlying random variables must have the same expected value. Focusing on the convex order first allows us to make comparison between reinsurance structures of the same "size." This is essentially the concept of risk defined by Rothschild and Stiglitz (1970) in economics. The standard characterizations for these stochastic orders are summarized as follows:

Proposition 2.1. *(Shaked and Shanthikumar 2007, theorems 41.A.3, 3.A.1 and 4.A.6) Let X and Y be two random variables. The stop-loss premium function of X is defined as $\pi_X(d) = \int_d^{\infty}(1 - F(x))dx$, where F is the distribution function and d is a stop-loss threshold.*

$$(1) X \preceq_{icx} Y \text{ if, and only if, } \pi_X(d) \leq \pi_Y(d), \forall d \geq 0$$
$$(2) \text{Given } E(X) = E(Y), X \preceq_{cx} Y \text{ if, and only if, } \pi_X(d) \leq \pi_Y(d), \forall d \geq 0$$
$$(3) X \preceq_{icx} Y \text{ if, and only if, there exist a random variable } Z \text{ such that } X \preceq_{st} Z \preceq_{cx} Y \text{ or } X \preceq_{cx} Z \preceq_{st} Y$$

Proposition 2.1 says that having a larger stop-loss premium is a necessary and sufficient condition for both the convex order and the increasing convex order. It can be shown that it is just a necessary condition for the usual stochastic order. Thus if $X \preceq_{st} Y$, then $X \preceq_{icx} Y$. Or equivalently in economics, if $-X$ is first-order stochastic dominating $-Y$, $-X$ is also second-order stochastic dominating $-Y$. Item (3) of the proposition above is the well-known separation theorem that links the three stochastic orders and will be used in Section 4 to show the dominance relationship between reinsurance structures with different expected loss costs.

Assuming equal means, a *sufficient* condition for one random variable having larger stop-loss premium than the other random variable for every stop-loss threshold is that the distribution functions of the two random variable cross only once.

Definition 2.4. Single Crossing Condition[3]: The cumulative distribution functions (CDF) F and G satisfy the single crossing condition if for some u^* in $(0, 1)$,

$$\begin{cases} F^{-1}(u) \leq G^{-1}(u) & \text{if } u \geq u^* \\ F^{-1}(u) \geq G^{-1}(u) & \text{if } u < u^*. \end{cases}$$

The following proposition shows that this single crossing property together with the equality of the means can be used to establish the convex order between two random variables.

Proposition 2.2. *(Rüschendorf 2013, Theorem 3.3.C; Denuit et al. 2005, Property 3.4.19) Let X and Y be two random variables with distribution functions F and G, respectively, such that $E(X) = E(Y)$. Then $X \preceq_{cx} Y$ if for some u^* in $(0, 1)$,*

$$\begin{cases} F^{-1}(u) \leq G^{-1}(u) & \text{if } u \geq u^* \\ F^{-1}(u) \geq G^{-1}(u) & \text{if } u < u^*. \end{cases}$$

We will see below in Section 3 that the comparison of the basic reinsurance structures can fit neatly into the framework with the single crossing condition. On the other hand, multiple crossings can happen between the distribution functions of other

[3]Also known as the Karlin-Novikov cut criterion in its simplest form or the thicker tail condition in the actuarial literature (Denuit et al. 2005).

types of reinsurance such as excess of loss reinsurance. To establish the ranking for those reinsurance structures, we need to use the property of closure under convolutions for the three stochastic orders *(Shaked and Shanthikumar 2007, theorems 1.A.3, 3.A.13 and 4.A.9)* as shown in sections 5 and 6.

3 Aggregate Loss Reinsurance

We first investigate three basic reinsurance structures: those with stop-loss, aggregate limit or quota share. The distribution functions for all these structures are defined on the same space as the gross aggregate loss ($0 \le S < \infty$), which is continuous and increasing. Assuming these structures have the same means, we'll show that they can be ranked using the convex order since their distribution functions cross only once when compared in pairs.

3.1 Three Basic Reinsurance Structures

Definition 3.1. Stop-Loss: The stop-loss reinsurance S_D with a threshold $D > 0$ is

$$S_D = \begin{cases} 0 & \textit{if } 0 \le S < D \\ S - D & \textit{if } D \le S. \end{cases}$$

Definition 3.2. Quota Share: Let $0 < q < 1$ be a quota share percentage. The quota share reinsurance is $S_q = qS$.

Definition 3.3. Aggregate Limit (i.e., 100 percent quota share with a cap): The reinsurance S_L with an aggregate limit $L > 0$ is

$$S_L = \begin{cases} S & \textit{if } 0 \le S < L \\ L & \textit{if } L \le S. \end{cases}$$

To illustrate the interrelationship of these reinsurance structures, the distribution functions F_{S_D}, F_{S_q} and F_{S_L} for the reinsurance contracts with stop-loss, quota share and aggregate limit, respectively, are graphed below in the typical Lee graph format (Lee 1988) with the y-axis as loss amount and the x-axis as distribution percentile. The area under each curve is the expected value of the respective loss random variable, which is assumed the same for all reinsurance structures in the illustration.

In figure 1, the blue curve is the distribution function for aggregate gross loss S while the red curve represents a stop-loss reinsurance S_D, which stays flat until the aggregate loss amount reaches the stop-loss threshold D at around 40^{th} percentile and then increases with the same incremental amounts as S. The expected retained loss amount by the cedant would be equivalent to the area between the two curves.

The green curve in figure 2 represents the distribution function for a reinsurance with an aggregate limit (S_L) while the red curve is for a stop-loss reinsurance (S_D).

The S_L curve follows the same path as the gross loss curve S and then becomes flat at the aggregate limit L. The areas between the curves before and after the intersection are the same and represent the trade-off between the two reinsurance structures. The curve for the stop-loss reinsurance is more spread out with higher weights in the upper tail.

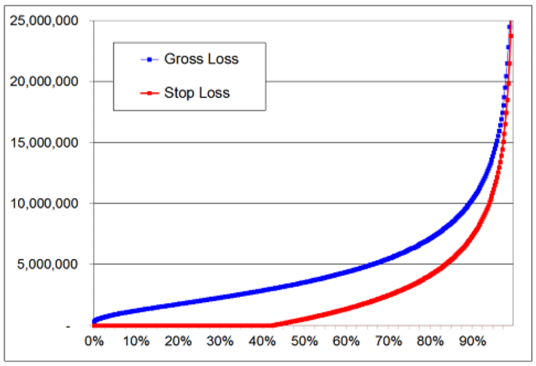

Figure 1

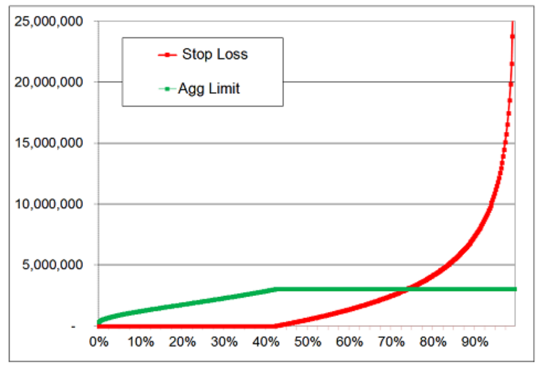

Figure 2

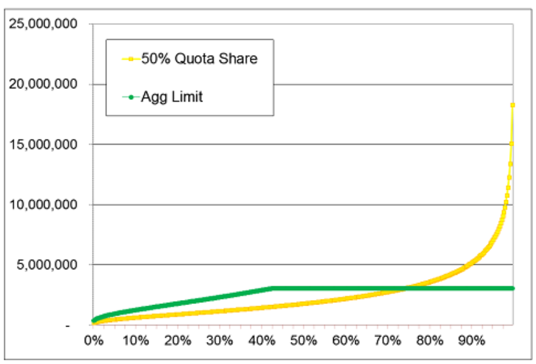

Figure 3

Figure 4

Figure 3 compares the curves between the aggregate limit reinsurance S_L and the quota share reinsurance S_q while figure 4 compares the latter with the stop-loss reinsurance S_D. Notice the differences between the curves in figures 3 and 4 are less than those in figure 2 as it will be shown later that the stop-loss reinsurance and aggregate limit reinsurance are the two extreme options in terms of riskiness.

The reason we can conveniently graph the distribution functions of S, S_D, S_L and S_q in the same space is that S_D, S_L and S_q are non-decreasing functions of S and are in fact comonotone (Denuit et al. 2005, Definition 1.9.1). That is, given a specific aggregate loss S^* and its percentile u^* on the distribution function of S, the corresponding S_D^*, S_L^* and S_q^* are all at the same percentile u^* on the distribution functions of S_D, S_L and S_q, respectively. This makes the comparison of reinsurance structures much more straightforward.

3.2 Risk Rankings of Basic Structures

The steps to show that stop-loss reinsurance dominates quota-share reinsurance follow the classical results in Van Heerwaarden, Kass and Goovaerts (1989), where they show that a risk-averse cedant would prefer the stop-loss reinsurance contract to all other contracts if all contracts have the same expected loss cost.

Proposition 3.1. *Assume that the stop-loss reinsurance S_D, the aggregate limit reinsurance S_L and the quota share reinsurance S_q for a ground-up aggregate loss S defined above have the same expected value. Under the convex order, the stop-loss reinsurance is more risky than the quota share reinsurance, which in turn is more risky than the reinsurance with an aggregate limit. That is $S_L \preceq_{cx} S_q \preceq_{cx} S_D$, or*

$$\textbf{Aggregate Limit} \preceq_{cx} \textbf{Quota Share} \preceq_{cx} \textbf{Stop-Loss}.$$

Proof. Based on Theorem 6.1 in Van Heerwaarden, Kass and Goovaerts (1989), the CDF of the retained loss net of a reinsurance with an aggregate deductible intersects only once with the CDF of the retained loss net of any other reinsurance structure given the equality of the mean losses. This also implies that the CDF of S_D crosses only once with the CDF of any other reinsurance structures including S_q. Let S^* denote the gross loss at the intersection of S_D and S_q. That is $S^* - D = qS^*$, or $S^* = \frac{D}{1-q}$. The value of S_D and S_q at the intersection would be $S_D^* = S_q^* = \frac{qD}{1-q}$ and $F_S(S^*) = F_{S_D}(\frac{qD}{1-q}) = F_{S_q}(\frac{qD}{1-q})$. Note that when $S^* < S$, $S_q < S_D$ since S_D increases faster than S_q. Similarly, $S_D \leq S_q$ when $S \leq S^*$. Thus the CDF of S_D crosses the CDF of S_q from below (in the context of a Lee graph). That is,

$$\begin{cases} F_{S_q}(x) \leq F_{S_D}(x) & \text{if } x \leq \frac{qD}{1-q} \\ F_{S_D}(x) \leq F_{S_q}(x) & \text{if } \frac{qD}{1-q} \leq x. \end{cases}$$

By Proposition 2.2, $S_q \preceq_{cx} S_D$ and the stop-loss reinsurance is more risky than the quota share reinsurance under the convex order. Similarly, we can demonstrate $S_L \preceq_{cx} S_q$. By the transitivity property of the convex order, $S_L \preceq_{cx} S_q \preceq_{cx} S_D$. □

Example 1: The XYZ Insurance Company writes \$50 million general liability insurance annually at an expected loss ratio of 70 percent. Currently the company has a 20 percent quota share treaty with a 20 percent ceding commission and no aggregate limit. XYZ is considering replacing the quota share with a stop-loss reinsurance treaty that attaches at an 80 percent loss ratio with a reinsurance premium of \$5.5 million. The company estimates that the expected loss cost of the stop-loss treaty is \$3.5 million, which means the implied margin is \$2 million.

Table 1 below shows the treaty premium, treaty expected loss and implied reinsurance margin for each structure.

Table 1 - Ranking Comparison

Option Type	Option Description	Insurance/Reins. Premium	Expected Loss Cost	Implied Margin
S	gross	\$50M	\$35M	
S_q	20% quota share	\$(10-2)M = \$8M	\$7M	\$1M
S_q	10% quota share	\$(5-1)M = \$4M	\$3.5M	\$0.5M
S_D	stop-loss	\$5.5M	\$3.5M	\$2M

Based on the risk-ordering analysis, the comparable treaties under the convex order are the stop-loss treaty and the 10 percent quota share treaty as both have the same expected loss cost and the former is more risky than the latter. This is also reflected in the extra margin charge of (2M−0.5M) = 1.5M. Note that the stop-loss threshold D is at an 80 percent loss ratio or \$40 million. The intersection point of the 10 percent quota share and the stop-loss treaties is at $S = D/(1-q) = 40/0.9 = 44.44$M or $S_D = S_q = qD/(1-q) = \$4.44$M. In other words, the 10 percent quota share treaty recovers more than the stop-loss treaty when the gross loss is less than \$44.44 million. The extra margin is meant to cover the uncertainty of the loss beyond \$44.44 million. The company should weigh their risk preference against the extra margin in selecting their reinsurance program.

3.3 Hybrid Reinsurance Structures

This line of reasoning and analysis can be extended to the aggregate loss reinsurance treaties with more than one contract feature, which include combinations of stop-loss, quota share and aggregate limit. As more features are included in a reinsurance structure, more parameters such as stop-loss threshold and aggregate limit need to be calibrated to make sure that the mean losses are the same across all treaties as required by the convex order. We define below two additional types of reinsurance and show that they can be properly ordered under the convex order along with the three basic reinsurance structures.

Definition 3.4. Quota Share with Aggregate Limit: The reinsurance $S_{q,L}$ with an aggregate limit $L > 0$ and a quota share percentage $0 < q < 1$ is

$$S_{q,L} = \begin{cases} qS & \textit{if } 0 \leq qS < L \\ L & \textit{if } L \leq qS. \end{cases}$$

Definition 3.5. Quota Share with Stop-Loss: The reinsurance $S_{D,q}$ with a stop-loss threshold $D > 0$ and a quota share percentage $0 < q < 1$ is

$$S_{D,q} = \begin{cases} 0 & \textit{if } 0 \leq qS < D \\ qS - D & \textit{if } D \leq qS. \end{cases}$$

Proposition 3.2. *Denote q, q_1 and q_2 as quota share percentages, D and D_2 as stop-loss thresholds and L and L_1 as aggregate limits. Let $0 < q < q_1 < 1$, $0 < q < q_2 < 1$, $0 < D_2 < D$ and $0 < L < L_1$ such that the reinsurance options, S_L, S_{q_1,L_1}, S_q, S_{D_2,q_2} and S_D for a ground-up aggregate loss S have the same expected value. That is,*

$$E(S_L) = E(S_{q_1,L_1}) = E(S_q) = E(S_{D_2,q_2}) = E(S_D).$$

Then the following orderings can be established:

$$S_L \preceq_{cx} S_{q_1,L_1} \preceq_{cx} S_q \preceq_{cx} S_{D_2,q_2} \preceq_{cx} S_D$$

or

$$\begin{aligned} &\textbf{Aggregate Limit} \\ \preceq_{cx} &\textbf{Mixture of Quota Share and Aggregate Limit} \\ \preceq_{cx} &\textbf{Quota Share} \\ \preceq_{cx} &\textbf{Mixture of Stop-Loss and Quota Share} \\ \preceq_{cx} &\textbf{Stop-Loss.} \end{aligned}$$

Proof. $S_L \preceq_{cx} S_{q_1,L_1}$ since the distribution function of S_{q_1,L_1} intersects only once with the distribution function of S_L from below at L and the single crossing condition applies. Similarly, since $0 < q < q_1 < 1$, the distribution function of S_q intersects only once with the distribution function of S_{q_1,L_1} from below at L_1 and thus $S_{q_1,L_1} \preceq_{cx} S_q$.

Since $0 < q < q_2 < 1$, S_{q_2} represents a larger layer than S_q. Similar to the proof in Proposition 3.1, the loss distribution function from a larger layer with a stop-loss such as S_{D_2,q_2} crosses only once with the distribution function of any other reinsurance option such as S_q, given that $E(S_q) = E(S_{D_2,q_2})$. And we conclude that $S_q \preceq_{cx} S_{D_2,q_2}$. The proof of $S_{D_2,q_2} \preceq_{cx} S_D$ follows the same argument in Proposition 3.1. □

Example 2: Continuing the example in Section 3.1, the XYZ Insurance Company considers lowering the stop-loss threshold from 80 percent to 75 percent loss ratio, but taking a 10 percent co-participation in the stop-loss treaty. It also considers adding an overall aggregate limit to the quota share reinsurance. It has been determined that both the new stop-loss reinsurance and a 12 percent quota share reinsurance with 9.6 million aggregate limit have an expected loss cost of 3.5 million.

Based on the risk-ranking analysis and the quotes received earlier in the example in Section 3.2, the reinsurance premium for the new stop-loss option should be between \$5.5 million and \$4 million and the premium for the 12 percent quota share reinsurance with a \$9.6 million aggregate limit should be less than \$4 million. In this case, Option S_L would be a 100 percent quota share reinsurance with a small overall aggregate limit such as \$5 million.

The new option S_{D_2,q_2} is obviously not the only treaty that can be ranked between the stop-loss reinsurance S_D and the quota share reinsurance S_q. Contract options can be created by decreasing the stop-loss threshold from the 80 percent loss ratio and reducing the quota share percentage from 100 percent such that the combination of the stop-loss threshold and quota share percentage have the same expected loss as before. Then in theory, an infinite number of options can be ordered and placed in between Options S_D and S_q. Under the convex order, the option with a higher stop-loss threshold would dominate those with lower stop-loss thresholds. Similarly, a continuum of options can fill the space between Options S_L and S_q by changing the quota share percentage and aggregate limit while keeping the expected loss cost constant.

Table 2 - Ranking Analysis

Option Type	Option Description	Quoted Reins. Premium	Expected Loss Cost	Implied Margin
S_L	100% quota share, \$5M aggregate limit		\$3.5M	
S_{q_1,L_1}	12% quota share, \$9.6M aggregate limit		\$3.5M	
S_q	10% quota share, No aggregate limit	\$(5-1)M=\$4M	\$3.5M	\$0.5M
S_{D_2,q_2}	Stop-loss attaching at 75% loss ratio, 90% quota share		\$3.5M	
S_D	Stop-loss attaching at 80% loss ratio, 100% quota share	\$5.5M	\$3.5M	\$2M

These hybrid reinsurance structures may seem like convex combinations of the three basic reinsurance structures. But, in fact, they are distinctively different. The hybrids represent non-linear trade-off between the basic reinsurance features, such as stop-loss thresholds, quota share percentages and aggregate limits. Although a 50 %/50% percent combination of S_q and S_D in the example above can be theoretically ranked between S_q and S_D, it is not a real option in reinsurance practice. S_{D_2,q_2}, on the other hand, does exist in practice with a higher quota share percentage than S_q and a lower stop-loss threshold than S_D.

4 Beyond Convex Order

We have shown in Section 3 that the basic reinsurance structures and their combinations can be compared in pairs and ranked using the convex order. The comparison is static in nature as the range of the structures is limited to those having the same expected loss cost. In this section, we expand the comparison to the structures with different expected loss costs. The tools that we use are the usual stochastic order and the increasing convex order as defined in Section 2. We show in the following proposition that the dominance relationship under these two stochastic orders for structures with different expected values can be clearly mapped out. On the other hand, some reinsurance structures are not comparable even though their expected loss costs may be far apart.

Proposition 4.1. *Denote q, q_1 and q_2 as quota share percentages, D and D_2 as stop-loss thresholds and L, and L_1 as aggregate limits. Let $0 < q < q_1 < 1$, $0 < q < q_2 < 1$, $0 < D_2 < D$ and $0 < L < L_1$ such that the reinsurance options, S_L, S_{q_1,L_1}, S_q, S_{D_2,q_2} and S_D for a ground-up aggregate loss S have the same expected value m. That is,*

$$E(S_L) = E(S_{q_1,L_1}) = E(S_q) = E(S_{D_2,q_2}) = E(S_D) = m.$$

Consider a similar set of reinsurance structures, $S_{L'}$, $S_{q_1',L_1'}$, $S_{q'}$, $S_{D_2',q_2'}$ and $S_{D'}$ for the same ground-up aggregate loss S, where $0 < q' < q_1' < 1$, $0 < q' < q_2' < 1$, $0 < D_2' < D'$ and $0 < L' < L_1'$ such that

$$E(S_{L'}) = E(S_{q_1',L_1'}) = E(S_{q'}) = E(S_{D_2',q_2'}) = E(S_{D'}) = n > m.$$

Then the following orderings can be established:

	$S_{L'}$	$S_{q_1',L_1'}$	$S_{q'}$	$S_{D_2',q_2'}$	$S_{D'}$
S_L	$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
S_{q_1,L_1}		$\preceq_{icx}$ *if* $q_1' < q_1, L_1 < L_1'$ $\preceq_{st}$ *if* $q_1 \le q_1', L_1 \le L_1'$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
S_q			$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$
S_{D_2,q_2}				$\preceq_{icx}$ *if* $q_2 < q_2', D_2 < D_2'$ $\preceq_{st}$ *if* $q_2 \le q_2', D_2' \le D_2$	$\preceq_{icx}$
S_D					$\preceq_{st}$

where the table reads, from left to right, $S_L \preceq_{st} S_{L'}$, $S_L \preceq_{icx} S_{q_1',L_1'}$, $S_{q_1,L_1} \preceq_{icx} S_{q_1',L_1'}$ if $q_1' < q_1$ and $L_1 < L_1'$, and so on.

Proof. We first show the usual stochastic orderings ($\preceq_{st}$) on the diagonal of the table above. Since $n > m$, the following inequalities must be true: $L < L'$, $q < q'$ and $D' < D$. Then the distribution functions of $S_{L'}$, $S_{q'}$ and $S_{D'}$ are above those of S_L, S_q and S_D, respectively, at every percentile. Thus we have $S_L \preceq_{st} S_{L'}$, $S_q \preceq_{st} S_{q'}$ and $S_D \preceq_{st} S_{D'}$. Similarly, if $q_1 \le q_1'$ and $L_1 \le L_1'$, the distribution function of $S_{q_1',L_1'}$ are above that of S_{q_1,L_1} at every percentile. By the same token, if $q_2 \le q_2'$ and $D_2' \le D_2$, the distribution function of $S_{D_2',q_2'}$ are above that of S_{D_2,q_2} at every percentile. This proves all the usual stochastic orderings ($\preceq_{st}$) on the diagonal.

Now we prove $S_{q_1,L_1} \preceq_{icx} S_{q_1',L_1'}$ if $q_1' < q_1$ and $L_1 < L_1'$. According to Proposition 3.2, we can find a $q^* < q_1$ such that $S_{q_1,L_1} \preceq_{cx} S_{q^*,L_1'}$. Since $E(S_{q_1',L_1'}) = n > m = E(S_{q^*,L_1'}) = E(S_{q_1,L_1})$, then $q^* < q_1'$ and $S_{q_1,L_1} \preceq_{cx} S_{q^*,L_1'} \preceq_{st} S_{q_1',L_1'}$. By Proposition 2.1, $S_{q_1,L_1} \preceq_{icx} S_{q_1',L_1'}$. Similarly, we can show $S_{D_2,q_2} \preceq_{icx} S_{D_2',q_2'}$ if $q_2 < q_2'$ and $D_2 < D_2'$.

Note that $S_L \preceq_{st} S_{L'} \preceq_{cx} S_{q_1',L_1'}$. By Proposition 2.1, $S_L \preceq_{icx} S_{q_1',L_1'}$. All the other increasing convex ordering pairs on the upper right corner of the table follow the same argument. □

Note that when $q_1 < q_1'$ and $L_1' < L_1$, no relationship can be derived between S_{q_1,L_1} and $S_{q_1',L_1'}$ as the former has a larger left tail while the latter has a larger right tail. Similarly, no ordering can be established for S_{D_2,q_2} and $S_{D_2',q_2'}$ when $q_2' < q_2$ and $D_2' < D_2$. Figures 5 and 6 illustrate this point.

In figure 5, the curve VV^* is the collection of reinsurance treaties with the same expected loss m, where each point on the curve represents a different combination of quota share percentage q_2 and stop-loss threshold D_2. For example, point V represents a reinsurance treaty with a \$50 million stop-loss threshold and a 60 percent quota share. Similarly, the curve $V'V''$ is the collection of reinsurance treaties, all having the same expected loss n, where $n > m$. Proposition 4.1 says that the relationship between the points on the $V'V''$ curve and the point V is such that the treaties above point V' on the $V'V''$ curve are riskier than V under the increasing convex order and the points between V' and V'' including V' and V'' are riskier than V under the usual stochastic order. The treaties below V'', however, do not have any dominating relationship with V.

Similarly in figure 6, the WW^* and the $W'W''$ curves represent the reinsurance structures having expected loss costs of m and n, respectively, where $n > m$. The treaties between W' and W'' are dominating W under the usual stochastic order while the treaties along the curve above W' are dominating W under the increasing convex order. No dominating relationship exists between W and those treaties below W'' on the $W'W''$ curve.

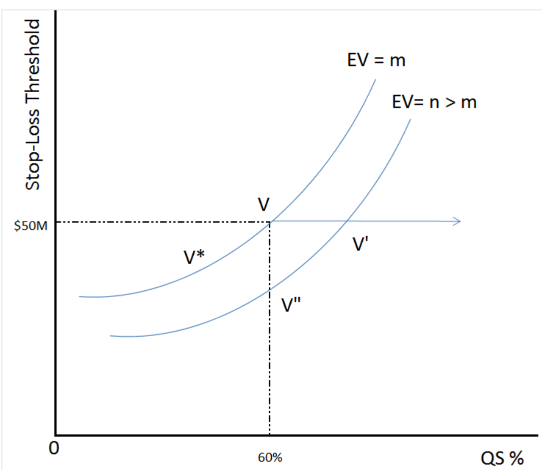

Figure 5

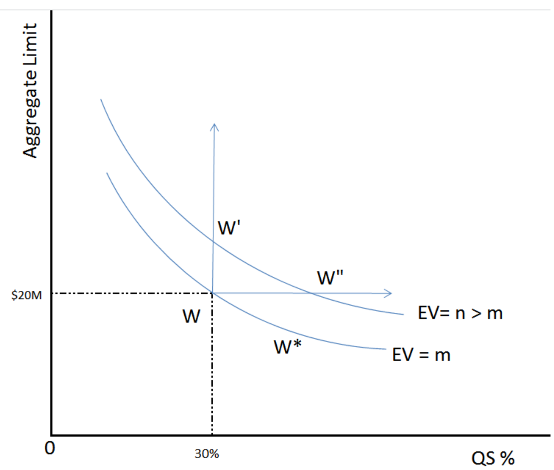

Figure 6

5 Application to Excess of Loss Reinsurance

The approach above can be applied to the excess of loss reinsurance except that the terminologies used in XOL are slightly different. The equivalent of a stop-loss threshold in an XOL reinsurance is called an aggregate deductible while the equivalent of a quota share in XOL is called partial placement or co-participation from a cedant's point of view. We first define the various XOL options.

5.1 Basic XOL Definitions and Risk Rankings

Definition 5.1. Excess of Loss: For $l, r > 0$, the (l xs r) layer loss for a risk or an occurrence is

$$(x-r)_+ \wedge l = \begin{cases} 0 & \textit{if } 0 \leq x < r \\ x-r & \textit{if } r \leq x < r+l \\ l & \textit{if } r+l \leq x. \end{cases}$$

Definition 5.2. Aggregate Layer Loss: Let $Y = \sum_{i=1}^{N}((x_i - r)_+ \wedge l)$ denote the aggregate layer loss for the (l xs r) layer where the summation is over the ground-up loss frequency random variable N with index i. $Y = 0$ when $N = 0$.

Definition 5.3. XOL with Aggregate Deductible: The XOL reinsurance Y_D with an aggregate deductible $D > 0$ is

$$Y_D = \begin{cases} 0 & \textit{if } 0 \leq Y < D \\ Y - D & \textit{if } D \leq Y. \end{cases}$$

Definition 5.4. XOL with Aggregate Limit: The XOL reinsurance Y_L with an aggregate limit $L > 0$ is

$$Y_L = \begin{cases} Y & \textit{if } 0 \leq Y < L \\ L & \textit{if } L \leq Y. \end{cases}$$

Definition 5.5. XOL with Partial Placement: Let $Y_q = qY$ denote the XOL reinsurance with partial placement where $0 < q < 1$ is the ratio ceded to reinsurers and $(1-q)$ is the cedant's co-participation ratio in the reinsurance.

Proposition 5.1. *Assume that the XOL reinsurance with aggregate deductible, aggregate limit and partial placement have the same expected value. Under the convex order, the XOL reinsurance with an aggregate deductible is more risky than the XOL reinsurance with partial placement, which in turn is more risky than the XOL reinsurance with an aggregate limit. That is* $Y_L \preceq_{cx} Y_q \preceq_{cx} Y_D$*, or*

$$\begin{aligned} &\textbf{XOL with Aggregate Limit} \\ \preceq_{cx}&\textbf{XOL with Partial Placement} \\ \preceq_{cx}&\textbf{XOL with Aggregate Deductible.} \end{aligned}$$

Proof. Similar to the proof of Proposition 3.1. □

Example 3: The XYZ Insurance Company writes \$100 million of commercial auto insurance annually. The company is presented with three reinsurance options: (1) \$4M xs \$1M XOL reinsurance with unlimited free reinstatements, (2) \$4M xs \$1M XOL reinsurance with an aggregate deductible of \$3 million and unlimited free reinstatements, or (3) \$4M xs \$1M XOL reinsurance with three free reinstatements. The company estimates that the expected loss costs for options 1, 2 and 3 are \$6 million, \$4 million and \$5.5 million, respectively and quoted reinsurance premiums are \$8 million, \$5.8 million and \$7 million, respectively.

Table 3 summarizes the estimated expected loss cost and market quotes for each of the reinsurance options:

Table 3 - Ranking Analysis

Option Type	Variation of 4x1 XOL	Quoted Reins. Premium	Expected Loss Cost	Implied Margin
Y	Free unlimited reinstatements	\$8M	\$6M	\$2M
Y_D	\$3M aggregate deductible	\$5.8M	\$4M	\$1.8M
Y_q	Free unlimited reinstatements 66.6% (= 4/6) placement	\$5.33M	\$4M	\$1.33M
Y_q	Free unlimited reinstatements 91.7% (= 5.5/6) placement	\$7.33M	\$5.5M	\$1.83M
Y_L	3 free reinstatements (aggregate limit = 16M)	\$7M	\$5.5M	\$1.5M

According to the risk-ranking analysis, a relevant comparison can be made between the 4x1 XOL reinsurance with a 66.6 percent placement and the 4x1 reinsurance with a \$3 million aggregate deductible as the expected loss costs are the same at \$4 million. The extra margin charge is \$0.47 million (1.8M–1.33M) for the risky aggregate deductible option. On the other hand, the theory indicates that the 4x1 XOL reinsurance with a 91.7 percent placement is more risky than the 4x1 XOL reinsurance with an aggregate limit of \$16 million (implied by the three reinstatements). The extra margin charge for the 4x1 XOL reinsurance with a 91.7 percent placement is \$0.33 million (1.83M–1.5M).

5.2 Higher XOL Retention as an Option

In XOL reinsurance, insurers can consider another option, namely adjusting their per risk/per occurrence retentions. Insurers often make these adjustments in response to changes in the underlying exposure and the implication to capital requirements. In this section, we will explore how an XOL reinsurance with a higher retention is stacking up against other types of XOL reinsurance in terms of risk ranking. Again we will assume all reinsurance structures under consideration in this section have the same expected value.

Definition 5.6. XOL With Higher Retention: Given an (l xs r) layer, the (l_H xs r_H) layer is a layer with a higher retention if $r < r_H$, $l_H < l$ and $(r + l) = (r_H + l_H)$. Let $Y_H = \sum_{i=1}^{N}((x_i - r_H)_+ \wedge l_H)$ denote the aggregate layer loss for the (l_H xs r_H) layer where the summation is over the ground-up loss x with frequency N.

For example, by definition, a \$3M xs \$2M XOL layer is a higher layer than a \$4M xs \$1M XOL layer while the sums of the respective limits and retentions are identical at \$5 million. Suppose the cedant co-participates in the \$4M xs \$1M layer so that the resulting \$4M xs \$1M XOL reinsurance with partial placement has the same expected value as the \$3M xs \$2M XOL reinsurance. We will show in the following proposition that the latter is more risky than the former under the convex order.

Proposition 5.2. *(Higher Retention vs. Partial Placement) Let Y_q denote the (l xs r) XOL with partial placement and Y_H denote the (l_H xs r_H) XOL where $r_H > r$, $l_H < l$ and $r + l = r_H + l_H$. Assuming $E(Y_q) = E(Y_H)$, then under the convex order, Y_H is more risky than Y_q. That is $Y_q \preceq_{cx} Y_H$, or*

$$\textbf{Partial Placement } \preceq_{cx} \textbf{Higher Retention}.$$

Proof. First we analyze the two per risk/occurrence severity random variables, $q[(x-r)_+ \wedge l]$ and $[(x-r_H)_+ \wedge l_H]$. The relationship between $q[(x-r)_+ \wedge l]$ and $[(x-r_H)_+ \wedge l_H]$ is similar to that of a quota share reinsurance with q as the quota share percentage and a stop-loss reinsurance with $(r_H - r)$ as the stop-loss threshold. Note that $qE[(x-r)_+ \wedge l] = E[(x-r_H)_+ \wedge l_H]$. Then the single crossing condition and the equality of the means imply that on the individual severity distribution basis,

$$q[(x-r)_+ \wedge l] \preceq_{cx} [(x-r_H)_+ \wedge l_H].$$

That is, under the convex order $[(x-r_H)_+ \wedge l_H]$ is more risky than $q[(x-r)_+ \wedge l]$. Note that $Y_q = \sum_{i=1}^{N} q[(x_i - r)_+ \wedge l]$ and $Y_H = \sum_{i=1}^{N} [(x - r_H)_+ \wedge l_H]$ where N is the number of risks/occurrences and $E(Y_q) = E(Y_H)$. As the convex order is closed under convolution (Shaked and Shanthikumar 2007, Theorem 3.A.13) and the frequency random variable N is independent, we get $Y_q \preceq_{cx} Y_H$.

□

Proposition 5.3. *(Higher Retention vs. Aggregate Deductible) Let Y_D denote the (l xs r) XOL reinsurance with aggregate deductible D and let Y_H denote the (l_H xs r_H) XOL reinsurance where $r < r_H$, $l_H < l$ and $(r + l) = (r_H + l_H)$. Assuming $E(Y_D) = E(Y_H)$, then under the convex order, Y_D is more risky than Y_H. That is $Y_H \preceq_{cx} Y_D$, or*

$$\textbf{XOL with Higher Retention } \preceq_{cx} \textbf{XOL with Aggregate Deductible.}$$

Proof. Similar to the proof for Proposition 3.1, Theorem 6.1 in Van Heerwaarden, Kass and Goovaerts (1989) implies that the CDF of Y_D also crosses only once with the CDF of any other XOL reinsurance structures such as Y_H given that the (l_H xs r_H) layer is a subset of the original layer. Given the equality of the means and the single crossing property, Proposition 2.2 implies that $Y_H \preceq_{cx} Y_D$.

□

Combining propositions 5.1, 5.2 and 5.3 and using the transitivity of the convex order, we obtain the following result:

$$\begin{aligned} &\textbf{XOL with Aggregate Limit} \\ \preceq_{cx}&\textbf{XOL with Partial Placement} \\ \preceq_{cx}&\textbf{XOL with Higher Retention} \\ \preceq_{cx}&\textbf{XOL with Aggregate Deductible.} \end{aligned}$$

The red curve in figure 7 represents the distribution function for a reinsurance with aggregate deductible Y_D, which stays flat until the aggregate layer loss amount reaches the deductible threshold D at the 25^{th} percentile and then increases with the same incremental amounts as Y.

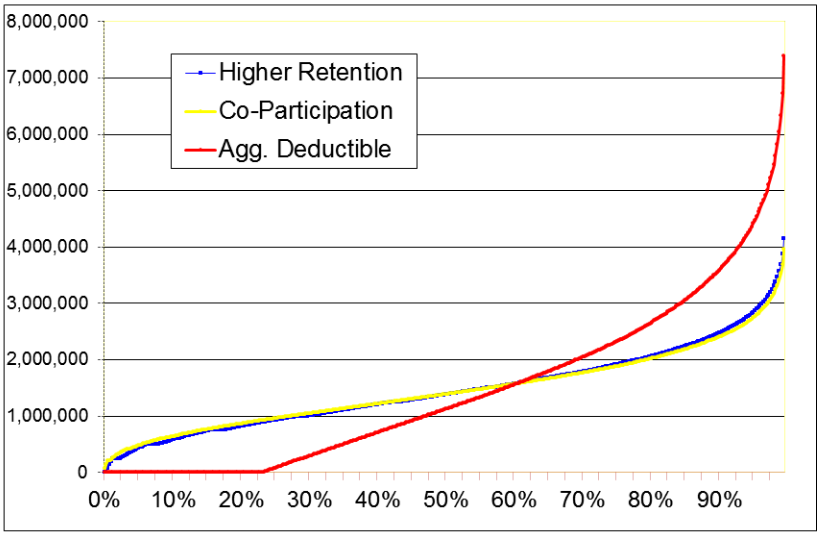

Figure 7

The yellow curve represents the distribution function of an XOL reinsurance with partial placement Y_q while the blue curve is for an XOL reinsurance Y_H with a per risk/occurrence retention level higher than that for Y. Given that Y_q and Y_H have the same expected loss, the blue Y_H curve starts under the yellow Y_q curve, then the two curves intertwine over most of the percentiles and finally the Y_H curve takes over after the last intersection at the 70^{th} percentile. Notice that the convex order does allow multiple crossings of the CDF curves as long as the stop-loss premium requirement in Proposition 2.1 is satisfied. When multiple crossings occur, it is difficult to discern convex order dominance empirically. Thus the analytical proof is an important confirmation of the dominance relationship and serves as an indication tool for reinsurance pricing.

5.3 Hybrid XOL Structures

Again this line of reasoning and analysis can be extended to the XOL treaties with more than one contract feature, which include combinations of aggregate deductibles, higher retentions, partial placement and/or aggregate limits. For common reinsurance structures with at most two contract features, proving stochastic ordering may be straightforward. We define below three additional types of reinsurance and show that they can be properly ordered under the convex order along with the four basic XOL reinsurance structures. For these reinsurance structures with two contract features, the proof of stochastic ordering is similar to those in propositions 5.2 and 5.3.

Definition 5.7. Mixture of Partial Placement and Aggregate Limit: The XOL reinsurance $Y_{q,L}$ with an aggregate limit $L > 0$ and a placement ratio $0 < q < 1$ is

$$Y_{q,L} = \begin{cases} qY & \text{if } 0 \leq qY < L \\ L & \text{if } L \leq qY. \end{cases}$$

Definition 5.8. Mixture of Higher Retention and Partial Placement: The XOL reinsurance $Y_{H,q}$ with a placement ratio $0 < q < 1$ and a higher retention as defined in Definition 5.6 is $Y_{H,q} = qY_H$.

Definition 5.9. Mixture of Aggregate Deductible and Higher Retention: The XOL reinsurance $Y_{D,q}$ with an aggregate deductible $D > 0$ and a higher retention as defined in Definition 5.6 is

$$Y_{H,D} = \begin{cases} 0 & \text{if } 0 \leq Y_H < D \\ Y_H - D & \text{if } D \leq Y_H. \end{cases}$$

Similar to Proposition 4.1, the following proposition uses the usual stochastic order and the increasing convex order and extends the analysis to include XOL reinsurance structures with different expected losses.

Proposition 5.4. *Denote H, H_2 and H_3 as higher retention layers, q, q_2 and q_3 as placement ratios, D and D_3 as aggregate deductibles and L and L_1 as aggregate limits. Let $0 < q < q_1 < 1$, $0 < q < q_2 < 1$, $0 < D_3 < D$, $0 < L < L_1$ and H be a higher layer than either H_2 or H_3 such that the XOL reinsurance options, Y_L, Y_{q_1,L_1}, Y_q, Y_{H_2,q_2}, Y_H, Y_{H_3,D_3} and Y_D for an aggregate layer loss Y have the same expected value, m. That is,*

$$E(Y_L) = E(Y_{q_1,L_1}) = E(Y_q) = E(Y_{H_2,q_2}) = E(Y_H) = E(Y_{H_3,D_3}) = E(Y_D) = m.$$

Then the following orderings can be established:

$$Y_L \preceq_{cx} Y_{q_1,L_1} \preceq_{cx} Y_q \preceq_{cx} Y_{H_2,q_2} \preceq_{cx} Y_H \preceq_{cx} Y_{H_3,D_3} \preceq_{cx} Y_D$$

or

XOL with Aggregate Limit
$\preceq_{cx}$**XOL with Mixture of Partial Placement and Aggregate Limit**
$\preceq_{cx}$**XOL with Partial Placement**
$\preceq_{cx}$**XOL with Mixture of Higher Retention and Partial Placement**
$\preceq_{cx}$**XOL with Higher Retention**
$\preceq_{cx}$**XOL with Mixture of Aggregate Deductible and Higher Retention**
$\preceq_{cx}$**XOL with Aggregate Deductible.**

Moreover, consider a similar set of reinsurance structures, $Y_{L'}$, $Y_{q_1',L_1'}$, $Y_{q'}$, $Y_{H_2',q_2'}$, $Y_{H'}$, $Y_{H_3',D_3'}$ and $Y_{D'}$, on the same layer aggregate loss Y, where $0 < q' < q_1' < 1$,

$0 < q' < q_2' < 1$, $0 < D_3' < D'$, $0 < L' < L_1'$ and H' is a higher layer than either H_2' or H_3' such that

$$E(Y_{L'}) = E(Y_{q_1',L_1'}) = E(Y_{q'}) = E(Y_{H_2',q_2'}) = E(Y_{H'}) = E(Y_{H_3',D_3'}) = E(Y_{D'}) = n > m.$$

Then the following orderings can be established:

	$Y_{L'}$	$Y_{q_1',L_1'}$	$Y_{q'}$	$Y_{H_2',q_2'}$	$Y_{H'}$	$Y_{H_3',D_3'}$	$Y_{D'}$
Y_L	$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_{q_1,L_1}		*see* *	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_q			$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_{H_2,q_2}				*see* **	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_H					$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_{H_3,D_3}						*see* ***	$\preceq_{icx}$
Y_D							$\preceq_{st}$

where the table reads, from left to right, $Y_L \preceq_{st} Y_{L'}$, $Y_L \preceq_{icx} Y_{q_1',L_1'}$ and so on.

*	$Y_{q_1',L_1'}$
Y_{q_1,L_1}	$\preceq_{icx}$ *if* $q_1' < q_1, L_1 < L_1'$ $\preceq_{st}$ *if* $q_1 \le q_1', L_1 \le L_1'$

**	$Y_{H_2',q_2'}$	***	$Y_{H_3',D_3'}$
Y_{H_2,q_2}	$\preceq_{icx}$ *if* $H_2 < H_2', q_2 < q_2'$ $\preceq_{st}$ *if* $H_2' \le H_2, q_2 \le q_2'$	Y_{H_3,D_3}	$\preceq_{icx}$ *if* $H_3' < H_3, D_3 < D_3'$ $\preceq_{st}$ *if* $H_3' \le H_3, D_3' \le D_3$

where in general $H < H'$ means H' has a higher per risk/occurrence retention than H.

Proof. The proof of $Y_L \preceq_{cx} Y_{q_1,L_1} \preceq_{cx} Y_q$ is similar to the proof of Proposition 3.2 while the proof of $Y_q \preceq_{cx} Y_{H_2,q_2} \preceq_{cx} Y_H$ is similar to the proof of Proposition 5.2, where the convex order is established first at the per risk/occurrence level. Use the closure by convolution property to prove the ordering at the aggregate layer loss level. Similarly, use Proposition 5.3 to prove $Y_H \preceq_{cx} Y_{H_3,D_3} \preceq_{cx} Y_D$ since $D_3 < D$ and H is a higher layer than H_3, which in turn is a higher layer than the original layer for Y_D.

The usual stochastic orderings ($\preceq_{st}$) and the increasing convex orderings ($\preceq_{icx}$) in the large table above are similar to those in Proposition 4.1 except the relationship for the structures with the higher retention layers. We need to show $Y_H \preceq_{st} Y_{H'}$ and the relationships in the (**) grid and (***) grid. Since $n > m$, H is a higher layer than H'. Then the distribution function of $Y_{H'}$ must be above that of Y_H at every percentile, hence $Y_H \preceq_{st} Y_{H'}$.

If $H_2' \le H_2$ and $q_2 \le q_2'$, the distribution function of $Y_{H_2',q_2'}$ must be above that of Y_{H_2,q_2} at every percentile. By the same token, if $H_3' \le H_3$ and $D_3' \le D_3$, the distribution function of $Y_{H_3',D_3'}$ must be above that of Y_{H_3,D_3} at every percentile. This proves all the usual stochastic ordering on the diagonal.

If $q_2 < q_2'$ and $H_2 < H_2'$, based on the first half of this proposition, we can find a q^* greater than q_2 such that $Y_{H_2,q_2} \preceq_{cx} Y_{H_2',q^*}$. Since $E(Y_{H_2',q_2'}) = n > m = E(Y_{H_2',q^*}) = E(Y_{H_2,q_2})$, then q^* must be smaller than q_2' and $Y_{H_2,q_2} \preceq_{cx} Y_{H_2',q^*} \preceq_{st} Y_{H_2',q_2'}$. By Proposition 2.1, $Y_{H_2,q_2} \preceq_{icx} Y_{H_2',q_2'}$. Similarly, we can show $Y_{H_3,D_3} \preceq_{icx} Y_{H_3',D_3'}$ if $H_3' < H_3$ and $D_3 < D_3'$ and $Y_{q_1,L_1} \preceq_{icx} Y_{q_1',L_1'}$ if $q_1' < q_1$ and $L_1 < L_1'$. □

Similar to figures 5 and 6 for aggregate loss reinsurance, figures 8 and 9 illustrate the relationships between $Y_{H_2',q_2'}$ and Y_{H_2,q_2} and between $Y_{H_3',D_3'}$ and Y_{H_3,D_3}, respectively. In figure 8, the curve VV^* represents the collection of reinsurance treaties with the same expected loss m, where each point on the curve is a different combination of placement percentage q_2 and layer retention H_2. Similarly, the curve $V'V''$ is the collection of reinsurance treaties, all having the same expected loss n, where $n > m$. Proposition 5.4 says that the relationship between the points on the $V'V''$ curve and the point V is such that the treaties above point V' on the $V'V''$ curve are riskier than V under the increasing convex order and the points between V' and V'' including V' and V'' are riskier than V under the usual stochastic order. The treaties below V'', however, do not have any dominating relationship with V. Similar interpretation can be made for figure 9, where the WW^* and the $W'W''$ curves represent reinsurance structures with different combinations of layer retentions and aggregate deductibles and having expected loss costs of m and n, respectively ($n > m$).

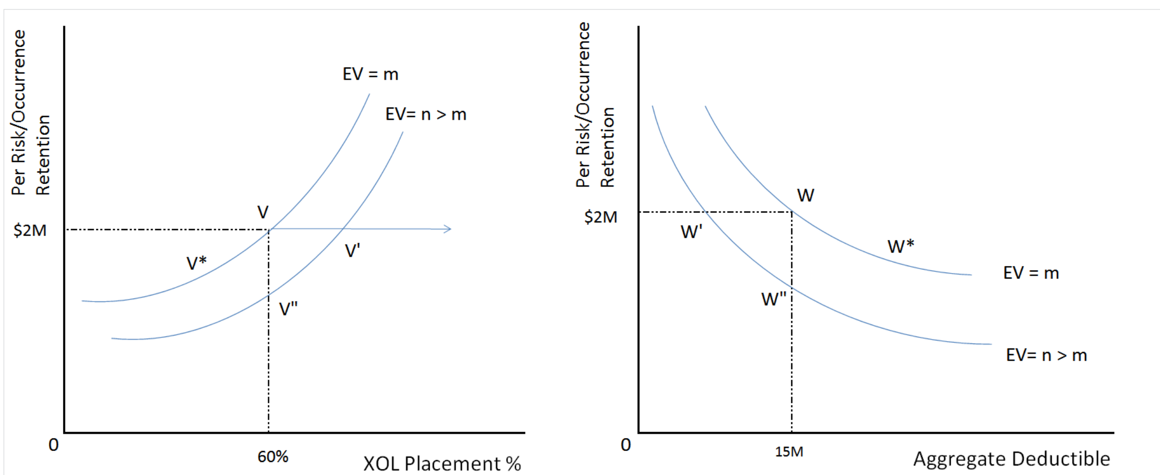

Figure 8 **Figure 9**

In general, with equal means, options with an aggregate deductible would dominate those without an aggregate deductible under the convex order. If both options have an aggregate deductible, then the one with a higher aggregate deductible would dominate the other with a lower aggregate deductible. Similarly, options without an aggregate limit would dominate those with an aggregate limit. If both options have an aggregate limit, then the one with a higher aggregate limit would dominate the other with a lower aggregate limit.

Example 4: Continuing the example in Section 5.1, the XYZ Insurance Company decides to explore other options by increasing the retention level of the XOL reinsurance for commercial auto liability and is willing to co-participate up to 20 percent. The company determines that the \$3M xs \$2M XOL reinsurance and the \$2.5M xs \$2.5M XOL reinsurance have the expected loss costs of \$5 million and \$4 million, respectively. Both options assume unlimited reinstatements. In addition, the company estimates that adding a \$1.5 million aggregate deductible to the \$3M xs \$2M XOL reinsurance can reduce the expected loss cost to \$4 million. Similarly, decreasing the number of free reinstatements from three to one for the \$4M xs \$1M XOL reinsurance also reduces the expected loss cost to \$4 million.

Based on the risk-ranking analysis and the quotes received earlier for the \$4M xs \$1M layer, the reinsurance premiums for these new options should be less than \$5.8 million and greater than \$5.33 million and should be in the order as shown in Table 4.

An interesting question can be raised as to how the reinsurance premium for an XOL layer (e.g., Option Y_H) can be approximated in general. Based on the risk-ranking results, one can find a premium lower bound for an XOL layer from a lower retention layer with partial placement (Option Y_{H_2,q_2}) and an upper bound from a lower retention layer with an aggregate deductible (Option Y_{H_3,D_3}). Clearly the closer the layers and the smaller the aggregate deductible, the better the approximation of the reinsurance premium.

Table 4 - Ranking Analysis

Option Type	Option Description	Quoted Reins. Premium	Expected Loss Cost	Implied Margin
Y_L	4x1 XOL, 100% placement, 1 free reinstatement		\$4M	
Y_{q_1,L_1}	4x1 XOL, 71.7% placement, 3 free reinstatements	\$5.09M	\$4M	\$1.09M
Y_q	4x1 XOL, 66.6% placement, free unlimited reinstatements	\$5.33M	\$4M	\$1.33M
Y_{H_2,q_2}	3x2 XOL, 80% placement, free unlimited reinstatements		(0.8*5) =\$4M	
Y_H	2.5x2.5 XOL, 100% placement, free unlimited reinstatements		\$4M	
Y_{H_3,D_3}	3x2 XOL, 100% placement, free unlimited reinstatements, \$1.5M aggregate deductible		\$4M	
Y_D	4x1 XOL, 100% placement, free unlimited reinstatements, \$3M aggregate deductible	\$5.8M	\$4M	\$1.8M

6 A Global Comparison

In reinsurance practice, the need to compare XOL reinsurance structures with the reinsurance on aggregate losses arises constantly. The metrics used in comparison are usually the distribution moments, such as mean and standard deviation along with some tail measures. In this section, we use the stochastic ordering approach to comparing reinsurance options that are on either an aggregate loss basis or an XOL basis.

Proposition 6.1. *Denote q as an XOL placement ratio, D and D_1 as a stop-loss threshold and an XOL aggregate deductible, respectively and L and L_1 as an aggregate limit and an XOL aggregate limit, respectively. Let $0 < q < 1$, $0 < D_1 < D$, $0 < L < L_1$ and H be a higher layer such that the reinsurance options, S_L, Y_{L_1}, Y_q, Y_H, Y_{D_1} and S_D have the same expected loss value, m. That is,*

$$E(S_L) = E(Y_{L_1}) = E(Y_q) = E(Y_H) = E(Y_{D_1}) = E(S_D) = m.$$

Then the following orderings can be established:

$$S_L \preceq_{cx} Y_{L_1} \preceq_{cx} Y_q \preceq_{cx} Y_H \preceq_{cx} Y_{D_1} \preceq_{cx} S_D$$

or

$$\begin{aligned} &\textbf{Aggregate Limit} \\ \preceq_{cx}&\textbf{XOL with Aggregate Limit} \\ \preceq_{cx}&\textbf{XOL with Partial Placement} \\ \preceq_{cx}&\textbf{XOL with Higher Retention} \\ \preceq_{cx}&\textbf{XOL with Aggregate Deductible} \\ \preceq_{cx}&\textbf{Stop-Loss.} \end{aligned}$$

Consider a similar set of reinsurance structures, $S_{L'}$, $Y_{L'_1}$, $Y_{q'}$, $Y_{H'}$, $Y_{D'_1}$ and $S_{D'}$, with regard to the same underlying loss, where $0 < q' < 1$, $0 < D'_1 < D'$ and $0 < L' < L'_1$ and H' is a higher per risk/occurrence layer such that

$$E(S_{L'}) = E(Y_{L'_1}) = E(Y_{q'}) = E(Y_{H'}) = E(Y_{D'_1}) = E(S_{D'}) = n > m.$$

Then the following orderings can be established:

	$S_{L'}$	$Y_{L'_1}$	$Y_{q'}$	$Y_{H'}$	$Y_{D'_1}$	$S_{D'}$
S_L	$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_{L_1}		$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_q			$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_H				$\preceq_{st}$	$\preceq_{icx}$	$\preceq_{icx}$
Y_{D_1}					$\preceq_{st}$	$\preceq_{icx}$
S_D						$\preceq_{st}$

where the table reads, from left to right, $S_L \preceq_{st} S_{L'}$, $S_L \preceq_{icx} Y_{L'_1}$, $Y_{L_1} \preceq_{st} Y_{L'_1}$ and so on.

Proof. The proof for the first half of the proposition follows the proofs in Proposition 3.1 and Proposition 5.4. The proof of the ($\preceq_{icx}$) and ($\preceq_{st}$) relationship in the grid follows the first half of this proposition and propositions 4.1 and 5.4, where we show that if $A \preceq_{cx} B \preceq_{st} C$, then $A \preceq_{icx} C$. □

Note that the quota share reinsurance is not compatible with this comparison framework involving XOL reinsurance. The right tail of the quota share reinsurance is always thicker than that of the XOL reinsurance while the opposite is true for the left tail. Proposition 6.1 also indicates that stop-loss reinsurance and the reinsurance with an aggregate limit serve as the upper and lower boundaries for the XOL reinsurance options. To make the comparison more complete, we can add the hybrid XOL reinsurance from Section 5.3. Given the transitivity of the convex order, the ranking of those hybrid XOL reinsurances would be the same as indicated in Proposition 5.3.

7 Implications to Pricing and Optimal Reinsurance

Assuming reinsurance companies are also risk averse, it is reasonable to assume that they would adopt premium principles that observe the established ordering above. Suppose the reinsurance structures under consideration have the same expected value and reinsurance companies employ the expected loss premium principle in calculation of the reinsurance premium. The actuarial literature indicates that stop-loss reinsurance would always be preferred by the cedant as it passes more risk to the reinsurer and costs the same as all the other options. Thus the implication of the risk ranking analysis above is that if reinsurance A is found to be more risky than reinsurance B, then reinsurance A should be priced higher than reinsurance B to compensate for the higher risk. As such, these ranking results may serve as an elementary tool in identifying inconsistent market quotes.

In the optimal reinsurance literature (e.g., Cheung 2010), the frequently used approach in finding optimal reinsurance is by maximization/minimization of an objective function over a convex constraint. The convex objective function (to be minimized) can be Value at Risk (VaR) or Tail Value at Risk (TVaR) of the retained exposure, which is defined as total exposure minus ceded exposure plus the reinsurance premium for the ceded exposure.

Definition 7.1. VaR Objective Function: The VaR objective function is

$$\mathbf{MinVaR}[X - f(X) + PR(f(X))]$$

where $f(X)$ is the ceded loss and $PR(f(X))$ is the corresponding reinsurance premium.

Obviously if the premium calculation is expected value based, the optimal reinsurance would always be the stop-loss reinsurance given that a tail measure is the selection criterion. Thus it is more realistic if the premium principle is convex in the maximization/minimization process (e.g., Chi 2012; Guerra and Centeno 2010).

The standard deviation principle and the variance principle along with the Wang principle are known to observe the second order stochastic dominance relationship. It would be interesting to evaluate the pricing differentials among the reinsurance structures using the three premium principles, which could be a subject for future research.

8 Conclusions

Reinsurance can be regarded as financial derivatives on a random loss process, which determines how reinsurers and insurers would share the loss upon its realization. The major technical difference between reinsurance and other financial derivatives such as stock options is that common reinsurance structures are comonotone with the underlying loss process. This makes the comparison of reinsurance structures intuitive and sometimes straightforward.

Following the classical results on optimal reinsurance in the actuarial literature, the paper[4] has shown that many common reinsurance structures in practice can be ranked either under the convex order if they have the same expected loss costs or under the increasing convex order and the usual stochastic order if they have different expected loss costs. Using the results of the paper, actuaries and underwriters can easily compare the riskiness of various reinsurance structures in an ERM and/or reinsurance retention analysis. The results also imply that reinsurers should price these reinsurance contracts with premium principles that recognize the risk rankings established in the paper.

[4]Loss discount and other accounting treatments that may be associated with specific reinsurance treaties are not considered here and are beyond the scope of this paper.

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