Probability of up-crossing before ruin for a Lévy process having two sided jumps

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Abstract

In this review work, we study in details the theorem (formula) for probability of up-crossing before down-crossing (or ruin in case of lower barrier is 0) by a Lévy process, insurance reserve, having both sided jumps given by Asmussen and Albrecher [5]. Here, we provide the details proof of the theorem as well as investigate it by using numerical example.

1 Introduction

A real-valued stochastic process $\{R_t : t \geq 0\}$ is said to be Lévy process if: (i) $R_0 = 0$ almost surely (a.s.), (ii) The increments are independent, i.e. for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$ $R_{t_2} - R_{t_1}, \quad R_{t_3} - R_{t_2}, \cdots, R_{t_n} - R_{t_{n-1}}$, (iii) For any $s < t$, $R_t - R_s \stackrel{\mathcal{D}}{=} R_{t-s}$, i.e. the increments are stationary and (iv) For any $\epsilon > 0$ and $t \geq 0$, $\lim_{h\to 0} \mathbb{P}\left(|R_{t+h} - R_t| > \epsilon\right) = 0$ i.e. continuous in probability.

The reserve of an insurer with initial capital $u \in [b, a]$, where a and b are the upper boundary and lower boundary respectively and having both sided jumps can be expressed by the following equation

$$R_t = u + \sum_{i=1}^{N_t^1} p_i - \sum_{i=1}^{N_t^2} c_i + \mu t + \sigma W_t \quad with \quad R_0 = u \tag{1}$$

where positive jumps $\{p_n\}_{n\geq 1}$ are a family of i.i.d. having distribution F_p and occur at the epochs of the Poisson(λ_p) process N_t^1 also independent of N_t^1 and is of phase-type with representation $(\alpha_p, \mathbf{T}_p)$, and negative jumps $\{c_n\}_{n\geq 1}$ are a family of i.i.d. having distribution F_c and occur at the epochs of the Poisson(λ_c) process N_t^2 also independent of N_t^2 and is of phase-type with representation $(\alpha_c, \mathbf{T}_c)$. μ is the drift of the Brownian motion and W_t is a standard Brownian motion with constant variance $\sigma^2 > 0$. The term

$\mu t + \sigma W_t$ represents the fluctuations in the money flow of the company, for example number of clients may change or other market fluctuations. Then equation(1) satisfies all four conditions of Lévy process.

Additionally, Lévy process has some special properties some of those are: (i) Lévy process can have two types of jumps, finitely many big jumps in unit time interval and infinitely many small jumps in unite time interval, (ii) if there is a positive measure, $\nu(dx)$, centred at $\mathbb{R} \setminus \{0\}$ satisfying $\int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(dx) < \infty$ (usually called Lévy measure), then jumps of Lévy process can be characterized by its Lévy measure $\nu(dx)$ [5], (iii) Lévy process can be decomposed as an independent sum of Brownian motion and compound Poisson like processes ([5],[6], [7]), (iv) Lévy process has cádlág (right continuous with left limit) path with finite variation in finite intervals ([5],[6], [7]), (v) Lévy process holds infinitely divisible property [2] and (vi) Every Lévy process is a semi-martingale [**?**].
According to property (iii) and using renowned Lévy- It$\hat{o}$ decomposition, we can decompose our reserve process. Lévy-It$\hat{o}$ decomposition [7] says that if there is a Lévy measure $\nu(dx)$, then the characteristic exponent also known as Lévy exponent $\mathcal{K}(\cdot)$ (defined by $\mathbb{E}(e^{\gamma R_t}) = e^{-t\mathcal{K}(\gamma)}$, for all $\gamma \in \mathbb{C}$) of an infinitely divisible process can be written

$$\mathcal{K}(\gamma) = \left\{\mu\gamma + \frac{\sigma^2\gamma^2}{2}\right\} + \left\{\nu(\mathbb{R} \setminus (-1,1)) \int_{|x|\geq 1} (1 - e^{\gamma x}) \frac{\nu(dx)}{\nu(\mathbb{R} \setminus (-1,1)}\right\} + \left\{\int_{0<|x|<1} (1 - e^{\gamma x} + \gamma x)\nu(dx)\right\}$$

Or equivalently,

$$\mathcal{K}(\gamma) = \mathcal{K}^1(\gamma) + \mathcal{K}^2(\gamma) + \mathcal{K}^3(\gamma) \tag{2}$$

for all $\gamma \in \mathbb{C}$, where $\mu, \sigma \in \mathbb{R}$. Moreover, $\mathcal{K}^1(\gamma)$ is the characteristic exponent of a linear Brownian motion, $\mathcal{K}^2(\gamma)$ is the characteristic exponent of an independent compound Poisson process with rate $\nu(\mathbb{R} \setminus (-1,1))$ having i.i.d. entries with common distribution $\frac{\nu(dx)}{\nu(\mathbb{R}\setminus(-1,1))}$ which are concentrated on $\{x : |x| \geq 1\}$ and $\mathcal{K}^3(\gamma)$ is the characteristic exponent of a square-integrable martingale. So, our Lévy (reserve) process can be decomposed as $R_t = R_t^1 + \underbrace{(R_t^p + R_t^c)}_{R_t^2} + R_t^3$, where R_t^1 is a linear Brownian motion, R_t^p and R_t^c are compound Poisson processes corresponding to premiums and claims respectively and R_t^3 is a square-integrable martingale with a.s. countable number of jumps or path of discontinuity on each finite time interval, which has magnitude less than unity. However, it is clear from the Lévy- Itô decomposition that $\mathcal{K}^1(\gamma), \mathcal{K}^2(\gamma)and\mathcal{K}^3(\gamma)$ are characteristic exponents of three different types of Lévy processes. Hence, $\mathcal{K}(\cdot)$ may be considered as the characteristic exponent of the independent sum of these three Lévy processes, by property (v) which is again a Lévy process. Therefore, the characteristic exponent of a Lévy process R_t can be defined as

$$\mathcal{K}(\gamma) = \mu\gamma + \frac{\sigma^2\gamma^2}{2} + \int_{\mathbb{R}} (1 - e^{\gamma x} + \gamma x \mathbb{1}_{(|x|<1)})\nu(dx) \tag{3}$$

for $\gamma \in \mathbb{C}$.
The function $\mathcal{K}(\gamma)$ completely determines the law of the process R_t.

Let us define the stopping times as follows:
$\tau_a = inf\{t \geq 0 : R_t \geq a\}$, $\tau_b = inf\{t \geq 0 : R_t \leq b\}$ and $\tau = \tau_a \wedge \tau_b$.
Let's present the process R_t in the form of a random walk: $R_n = u + S_n$, where $S_n = \sum_{i=0}^{n}(R_i - R_{i-1})$. Then we can use the following theorem:

Theorem 1.1. *[3] For a random walk on $\mathbb{R}$ there are only four possibilities, one of which has probability 1. (i) $S_n = u$ for all n (ii) $S_n \to \infty$ (iii) $S_n \to -\infty$ (iv) $-\infty = \lim\ \mathrm{inf} S_n < \lim\ \mathrm{sup} S_n = \infty$.*

We are not interested case (i) but the other cases ensure us that (if there are two boundaries) the process will attain either one or both boundaries. Hence, we have $\mathbb{P}_u(\tau = \tau_a) + \mathbb{P}_u(\tau = \tau_b) = 1$.
Throughout the paper we use $\mathbb{P}_u$ to denote the law of R_t such that $R_0 = u$ and $\mathbb{E}_u$ for corresponding expectation.
This literature is oriented in the following way: section 2 contains basics on phase-type distribution. In section 3 we discuss Lévy exponent of infinitely divisible process, compound Poisson distribution and phase-type distribution. Section 4 has some important martingales. In section 5 we bring exact formula for probability of up-crossing before down-crossing (or ruin) by a Lévy process. Section 6 contains an empirical example.

2 Phase type premiums and claims

Here we define the phase-type distribution. Everything in this section are taken from [1] by Ali and Pärna however, the terminologies and notations are based on [2] by Asmussen and Albrecher.

2.1 Phase type distribution

Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov chain with finitely many states denoted by $1, 2, \ldots, n, \Delta$. The state Δ is assumed to be absorbing and all other states are transient. The transition probability matrix of X_t is denoted by $\mathbf{P}$, the i^{th} row being the conditional distribution of the next state given the current state i. Let $\mathbf{T}$ denote the transition intensity matrix for the states $1, \ldots, n$. Then the intensity matrix (transition rate matrix, infinitesimal generator) for the whole Markov chain can be written in block-partitioned form as

$$\left(\begin{array}{c|c} \mathbf{T} & \mathbf{t} \\ \hline \mathbf{0} & 0 \end{array}\right)$$

where

$$\mathbf{t} = -\mathbf{T}\mathbf{e}$$

and $\mathbf{e} = (1, 1, \cdots, 1)'$. The vector $\mathbf{t}$ represents the *exit rate vector* with its i-th component t_i being the intensity of leaving the state i for the absorbing state Δ.

Definition 2.1. The distribution of the absorption time in the Markov chain described above is called phase type distribution.

Let $\boldsymbol{\alpha}$ be a row vector representing the initial distribution of states $1, 2, \ldots, n$. The couple $(\boldsymbol{\alpha}, \mathbf{T})$ is called the *representation* of the phase type distribution. The density of a phase type distribution can be written as

$$f(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{t}, \quad x \geq 0. \tag{4}$$

It is seen that phase type distribution is a generalization of the exponential distribution which corresponds to the case $n = 1$.

3 Lévy exponent of compound Poisson distribution

In this section at first, we define the Lévy exponent of a stochastic process then we present Lévy exponent of compound Poisson process that is compound Poisson process corresponding to our premium process and claim arrival process.

3.1 Lévy exponent

Cumulant generating function (c.g.f) for any $\gamma \in \mathbb{C}$ of a infinitely divisible process, $R_t, t \in \mathbb{R}$ having distribution F_{R_t} is of the form

$$\log(M_{R_t}(\gamma)) = \log \mathbb{E}[e^{\gamma R_t}] = \log \int_0^{+\infty} e^{\gamma x} dF_{R_t}(x) = t\mathcal{K}(\gamma).$$

Definition 3.1. The component $\mathcal{K}(\gamma)$ is known as Lévy exponent and define by

$$\mathcal{K}(\gamma) = \frac{1}{t} \log \mathbb{E}(e^{\gamma R_t}). \tag{5}$$

3.2 Lévy exponent of compound Poisson process

If X_t is of phase-type with representation $(\boldsymbol{\alpha}, \mathbf{T})$, then moment generating function of X_t is given by

$$M_{X_t}(\gamma) = \boldsymbol{\alpha}(-\gamma\mathbf{I} - \mathbf{T})^{-1}\mathbf{t} \tag{6}$$

If Z_t is a compound Poisson process denoted by $Z_t = \sum_{i=1}^{N_t} X_i$ where $N_t, t \geq 0$ is a Poisson process with rate λ and $\{X_n\}_{n\geq 1}$ is a family of i.i.d. random variables which are independent from $\{N_t, t \geq 0\}$ as well. Then m.g.f of Z_t is given by

$$M_{Z_t}(\gamma) = e^{\lambda t(M_{X_1}(\gamma)-1)} \tag{7}$$

Using equations (5), (6) and (7) we may define Lévy exponent of a compound Poisson process.

Definition 3.2. Lévy exponent of a compound Poisson process of aforementioned type is defined by

$$\mathcal{K}(\gamma) = \lambda(\boldsymbol{\alpha}(-\gamma\mathbf{I} - \mathbf{T})^{-1}\mathbf{t} - 1), \text{for any} \quad \gamma \in \mathbb{C} \tag{8}$$

According to equation (1), our premium and claim arrival processes can be defined as follows: $P = \sum_{i=1}^{N_t^1} p_i$ and $C = \sum_{i=1}^{N_t^2} c_i$. Premium and claim sizes, p_i and c_i are of phase-type with representation $(\boldsymbol{\alpha}_p, \mathbf{T}_p)$ and $(\boldsymbol{\alpha}_c, \mathbf{T}_c)$ respectively. Therefore, using (8) we can write the Lévy exponent of premium process and claim arrival process in the following way:

$$\mathcal{K}_P(\gamma) = \lambda_p \left(\boldsymbol{\alpha}_p \left(-\gamma\mathbf{I} - \mathbf{T}_p\right)^{-1} \mathbf{t}_p - 1\right) \tag{9a}$$

$$\mathcal{K}_C(\gamma) = \lambda_c \left(\boldsymbol{\alpha}_c \left(\gamma\mathbf{I} - \mathbf{T}_c\right)^{-1} \mathbf{t}_c - 1\right) \tag{9b}$$

Also, we know that Laplace exponent of Wiener (W) process with drift μ and constant variance $\sigma^2 > 0$ is

$$\mathcal{K}_W(\gamma) = \gamma\mu + \frac{\gamma^2\sigma^2}{2} \tag{9c}$$

Summing up sub-equations of (9), we obtain

$$\begin{aligned}\mathcal{K}(\gamma) = \gamma\mu + \frac{\gamma^2\sigma^2}{2} + \lambda_p \left(\boldsymbol{\alpha}_p \left(-\gamma\mathbf{I} - \mathbf{T}_p\right)^{-1} \mathbf{t}_p - 1\right) + \\ \lambda_c \left(\boldsymbol{\alpha}_c \left(\gamma\mathbf{I} - \mathbf{T}_c\right)^{-1} \mathbf{t}_c - 1\right)\end{aligned} \tag{10}$$

Proposition 3.1. Suppose the premiums and claims are of phase-type with representation $(\boldsymbol{\alpha}_p, \mathbf{T}_p)$ and $(\boldsymbol{\alpha}_c, \mathbf{T}_c)$ and their corresponding compound Poisson processes have jump rates λ_p and λ_c respectively. Then Laplace exponent of Lévy process (1) is given by (10) whenever $\mathcal{K}(\gamma)$ is well-defined for any $\gamma \in \mathbb{C}$.

In the next chapter we discuss two important martingales, namely Wald martingale and Kella-Whitt martingale and Doobs optional stopping time. The Kella-Whitt martingale for our Reserve process helps us to use Doobs optional stopping time. However we need Wald martingale to proof Kella-Whitt martingale.

4 Some important martingales and properties of martingale

In this section we define martingale, local martingale, some important martingales and some important properties of martingale. This chapter is based on Kella Whitt [8] and Asmussen [4].

4.1 Martingale

Definition 4.1. A stochastic process $\{Y_t\}_{t\geq 0}$ is said to be martingale with respect to a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, if

$$\mathbb{E}[Y_{t+s}|\mathcal{F}_t] = Y_t, \quad \forall s \leq t \tag{11}$$

Definition 4.2. A local martingale is a type of stochastic process, satisfying the localized version of the martingale property.

Furthermore, every martingale is a local martingale and every bounded local martingale is a martingale.

4.2 Wald martingale

Theorem 4.1. *If R_t is a Lévy process with Lévy exponent $\mathcal{K}(\gamma)$. Then*

$$M_t = e^{\gamma R_t - t\mathcal{K}(\gamma)} \tag{12}$$

is a martingale with respect to the filtration $\mathcal{F}_t$.

Proof. As $\mathbb{E}(e^{\gamma R_t}) = e^{t\mathcal{K}(\gamma)}$, we see that $\mathbb{E}(|M_t|) = \mathbb{E}(e^{\gamma R_t})\mathbb{E}(e^{-t\mathcal{K}(\gamma)}) < \infty \quad for \quad each \quad t \geq 0$.
For each $0 \leq s \leq t$, let us define

$$M_t = M_s e^{\gamma(R_t - R_s) - (t-s)\mathcal{K}(\gamma)}$$

Being Lévy process R_t has independent and stationary increments, i.e. $R_t - R_s \stackrel{\mathcal{D}}{=} R_{t-s}$. Moreover, M_s is $\mathcal{F}_s$ measurable, hence taking conditional expectation on the above expression with respect to the filtration $\mathcal{F}_s$, we have

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \mathbb{E}[e^{\gamma R_{t-s}} e^{-(t-s)\mathcal{K}(\gamma)}]$$

However, $\mathbb{E}(e^{\gamma R_t}) = e^{t\mathcal{K}(\gamma)}$ implies $\mathbb{E}e^{\gamma R_{t-s}} = e^{(t-s)\mathcal{K}(\gamma)}$. That is $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$. Therefore, M_t is a martingale. □

Definition 4.3. The martingale, M_t, defined in (12) is called Wald martingale.

4.3 Kella-Whitt martingale

The following definition and theorems are based on Kella-Whitt [9] and Protter [**?**].

Definition 4.4. The total variation of a real valued (or complex valued) function g, defined on $[a, b] \subset \mathbb{R}$ is defined by

$$V_a^b(g) = sup_P \sum_{i=0}^{n-1} |g(x_{i+1} - g(x_i)|,$$

where supremum considers set of all partitions, $P = \{a = x_0, x_1, \cdots, x_n = b\}$ of the given interval $[a, b]$.

Definition 4.5. The Lebesgue-Stieltjes differentials for a function g of bounded variation is defined by

$$dg(x) = dg_1(x) - dg_2(x),$$

where $g_1(x) = V_a^x(g)$ is the total variation of g in the interval $[a, x]$ and $g_2(x) = g_1(x) - g(x)$. Both g_1 and g_2 are monotono decreasing.

Cádlág process has only discontinuity at jumps [9],[?] and the jumps are finitely countable. Let define the discontinuous jumps at t by $\Delta J_t = J_t - J_{t^-}$, where $J_{t^-} = \lim_{s\uparrow t} J_s$ which helps to write the following definition. Moreover, if $\sup_t |\Delta J_t| \leq c < \infty$, for any constant c. Then we say J_t has bounded jumps.

Definition 4.6. The jumps of a Lévy process can be defined by

$$J_t = \int_0^t dJ_s^c + \sum_{0\leq s\leq t} \Delta J_s \tag{13}$$

that is $\{J_t^c\}_{t\geq 0}$ is a continuous adapted process with $J_0^c = 0$ and have bounded variation on finite intervals.

Theorem 4.2. *Let $\{R_t\}$ be a Lévy process with Lévy exponent $\mathcal{K}(\gamma)$, for all $\gamma \in \mathbb{C}$, let $\{J_t\}_{t\geq 0}$ be an adapted cádlág process of bounded variation on finite intervals defined in (13) and let $Z_t = R_t + J_t$. Then for each t, the random variable K_t defined by*

$$K_t = \mathcal{K}(\gamma)\int_0^t e^{\gamma Z_s} ds + e^{\gamma J_0} - e^{\gamma Z_t} + \gamma \int_0^t e^{\gamma Z_s} dJ_s^c + \sum_{0< s\leq t} e^{\gamma Z_s}\left(1 - e^{-\gamma \Delta J_s}\right) \tag{14}$$

is a local martingale whenever $\mathcal{K}(\gamma)$ is well-defined. Moreover, if if the expected variation of $\{J_t^c\}_{t\geq 0}$ and the expected number of jumps of $\{J_t\}_{t\geq 0}$ are finite on every finite intervals, then K_t, defined in (14) is a martingale.

Before giving proof of theorem (4.2) let us proof the following lemma.

Lemma 4.1. *Kella-Whitt martingale, K_t is uniformly convergent.*

Proof. If we consider $J_t = 0$ Kella-Whittt martingale given in (14) will reduce to

$$K_t = \mathcal{K}(\gamma)\int_0^{t\wedge\tau} e^{\gamma R_s} ds + e^{\gamma u} - e^{\gamma R_{t\wedge\tau}}, \gamma \in \mathbb{C}. \tag{15}$$

By considering $\tau = \tau_a \wedge \tau_b$ implies

$$|K_t| \leq |\mathcal{K}(\gamma)|\tau e^{|\gamma| max(|b|,a)} + e^{|\gamma|u} + e^{|\gamma|(a-u+V_p)} + e^{|\gamma|(u-|b|+V_c)}$$

where V_p represents the possible overshoot over a and V_c represents possible undershoot under b. Moreover, V_p and V_c are of phase-type with representation $(\mathbf{e}_i, \mathbf{T})$. Therefore, both $\mathbb{E}(e^{|\gamma|(a-u+V_p)})$ and $\mathbb{E}(e^{|\gamma|(u-|b|+V_c)})$ are finite for $0 \leq t \leq \tau$. Also, $\mathbb{E}\tau < \infty$. Therefore, $sup_{t\leq\tau}|K_t| < \infty$. That is K_t is uniformly convergent in other words it is integrable. □

Proof. of theorem (4.2). The proof is based on Kella-Whitt [8] and Protter [9]. Consider the Wald martingale $M_t = e^{\gamma X_t - t\mathcal{K}(\gamma)}$ and the process $B_t = e^{\gamma J_t + t\mathcal{K}(\gamma)}$, where J_t as in (13). Hence, with the help of stochastic integration by parts we get

$$M_tB_t - M_0B_0 = \int_0^t M_{s^-}dB_s + \int_0^t B_{s^-}dM_s + \sum_{0<s\leq t} \Delta M_s \Delta B_s \tag{16}$$

Since $\{B_t\}_{t\geq 0}$ is of bounded variation on bounded intervals, the last term of (16) is valid. However, $\int_0^t \Delta M_s dB_s = \sum_{0<s\leq t} \Delta M_s \Delta B_s$, So, we have

$$\begin{aligned} M_tB_t - M_0B_0 &= \int_0^t M_{s^-}dB_s + \int_0^t B_{s^-}dM_s + \int_0^t \Delta M_s dB_s \\ &= \int_0^t B_{s^-}dM_s + \int_0^t (M_{s^-} + \Delta M_s)dB_s \\ &= \int_0^t B_{s^-}dM_s + \int_0^t M_s dB_s \end{aligned}$$

That is

$$-\int_0^t B_{s^-}dM_s = \int_0^t M_s dB_s + M_0B_0 - M_tB_t \tag{17}$$

We know that $\{M_t\}_{t\geq 0}$ is a martingale therefore the left side of equation (17) is a local martingale which implies that the right side is a local martingale. The proof will conclude if we can identify the right side of (17) with K_t given in (14). For $0 < s \leq t$ and taking derivative of B_t we obtain

$$\begin{aligned} dB_s &= B_s\{\gamma dJ_s + \mathcal{K}(\gamma)ds\} \\ &= B_s\mathcal{K}(\gamma)ds + \gamma B_s dJ_s^c + B_s(\gamma \sum_{0\leq s\leq t} \Delta J_s) \\ &= B_s\mathcal{K}(\gamma)ds + \gamma B_s dJ_s^c + B_s(1 - e^{-\gamma \Delta J_s}) \end{aligned}$$

The last expression is Lebesgue-Stieltjes type and hence defined path by path. So, $sup_t|J_t^c| < c$ for any constant c. That is $\{J_t^c\}_{t\geq 0}$ has finite expected variation and $\{J_t\}_{t\geq 0}$ has finite expected number of jumps on every finite interval. Which implies $\mathbb{E}sup_{0\leq s\leq t}|K(t)| < \infty$ for every finite t. Hence by dominated convergence theory K_t is a martingale. □

It is known that Doob's optional stopping time is permissible for an integrable martingale which leads us to write the following definition.

4.4 Doobs optional stopping time

Proposition 4.1. If for a given t, $\mathbb{E}sup_{s\leq t}|K_s| < \infty$, then K_t , given in (14), is a proper martingale. Moreover, if τ be a stopping time such that $\mathbb{E}sup_{t\leq \tau}|K_t| < \infty$, then $\mathbb{E}K_\tau = \mathbb{E}K_0$.

In the next chapter we will see the application of Kella-Whitt martingale and Doob's optional stopping time theories on our reserve process (1).

5 Application on premiums and claims

In this chapter we bring exact formula for probability of up-crossing before down-crossing by a Lévy process having two sided jumps, both of the jumps are of phase-type. This chapter is based on Asmussen [4], [6] and Asmussen and Albrecher [5].

5.1 Probability of crossing boundaries by a Lévy process

Suppose Lévy exponent of (1) is given by (10). Let the event of crossing upper barrier a before lower barrier b resulting by a Brownian motion and not a jump denoted by V_0^p. Similarly, the event of crossing the lower barrier b before upper barrier a by a Brownian motion and not a jump be V_0^c. Moreover, let V_i^p illustrate the events of crossing a before b by a jump when the process is at phase i and V_i^c be the events of crossing b before a by a jump when the process is at phase i . Then both overshoot, V_p (value of R_t over a) and undershoot, V_c (value of R_t below b) are of phase-type with representations $(\mathbf{e}_i, \mathbf{T}_p)$ and $(\mathbf{e}_i, \mathbf{T}_c)$ respectively, where $\mathbf{e}_i$ is the i^{th} unit column vector, i.e. the i^{th} entry is 1 and all other are 0. Hence, their moment generating functions are $\mathbf{e}_i^{'}(-\gamma\mathbf{I} - \mathbf{T}_p)^{-1}\mathbf{t}_p$ and $\mathbf{e}_i^{'}(\gamma\mathbf{I} - \mathbf{T}_c)^{-1}\mathbf{t}_c$ respectively. Also, we denote eigenvalue with largest real part of $-\mathbf{T}_p$ be ρ^+ and eigenvalue with smallest real part of $\mathbf{T}_c$ be ρ^-.

Theorem 5.1. *If a phase-type distribution has n_1 transient state , then the corresponding intensity matrix is a square matrix of order $n_1 \times n_1$. Moreover, its moment generating function is a rational expression whose numerator is a polynomial of degree $n_1 - 1$ and denominator is a polynomial of degree n_1.*

Proof. First part of the proof is trivial. According to the definition of intensity matrix of a phase-type distribution, it is clear that the intensity matrix, $\mathbf{T}$ is a square matrix. In addition to that if there are n_1 transient states then order of

$\mathbf{T}$ is $n_1 \times n_1$. Moreover, we see that moment generating function of phase-type distribution is of the form $\underbrace{\boldsymbol{\alpha}}_{1\times n_1}\underbrace{(\gamma\mathbf{I}-\mathbf{T})^{-1}}_{n_1\times n_1}\underbrace{\mathbf{t}}_{n_1\times 1}$, as an example.The inverse matrix $\underbrace{(\gamma\mathbf{I}-\mathbf{T})^{-1}}_{n_1\times n_1}$ will contain entries of the form $\frac{c}{d(\gamma)}$, where c is a constant and $d(\gamma)$ is a polynomial of degree n_1. In addition to that $\boldsymbol{\alpha}$ is a row vector and $\mathbf{t}$ is a column vector of constants. So, after simplifying the m.g.f. of phase-type distribution, $\boldsymbol{\alpha}(\gamma\mathbf{I}-\mathbf{T})^{-1}\mathbf{t}$, we will obtain a rational expression of the form $\frac{n(\gamma)}{d(\gamma)}$ where $n(\gamma)$ is a polynomial of degree $n_1 - 1$ and $d(\gamma)$ is a polynomial of degree n_1. □

Corollary 5.1. *Suppose premium size has phase-type distribution with n_p number of transient states and claim size has phase-type distribution with n_c number of transient states. Then there exist $n = n_p + n_c + 2$ distinct complex numbers γ_n such that Cramér-Lundberg equation holds, i.e. $\mathcal{K}(\gamma_i) = 0, i = 1, 2, \cdots, n$.*

Proof. According to theorem (5.1) it is obvious that moment generation function of overshoot ($\alpha_p(\gamma\mathbf{I}-\mathbf{T}_p)^{-1}\mathbf{t}_p$) corresponding to premiums is a rational expression of the form $\frac{n_p(\gamma)}{d_p(\gamma)}$, where $n_p(\gamma)$ is a polynomial of degree $n_p - 1$ and $d_p(\gamma)$ is a polynomial of degree n_p. Similar argument is applicable for $\alpha_c(\gamma\mathbf{I}-\mathbf{T}_c)^{-1}\mathbf{t}_c$, i.e. it will be a polynomial of degree $\frac{n_c(\gamma)}{d_c(\gamma)}$ with $n_c(\gamma)$ is a polynomial of degree $n_c - 1$ and $d_c(\gamma)$ is a polynomial of degree n_c. Then using (10) Cramér-Lundberg equation $\mathcal{K}(\gamma_i) = 0$ can be written as follows:

$$0 = \gamma\mu + \frac{\gamma^2\mu^2}{2} + \lambda_p\left(\frac{n_p(\gamma)}{d_p(\gamma)} - 1\right) + \lambda_c\left(\frac{n_c(\gamma)}{d_c(\gamma)} - 1\right)$$

i.e.

$$\begin{aligned} 0 = d_p(\gamma)d_c(\gamma)\gamma\mu + d_p(\gamma)d_c(\gamma)\frac{\gamma^2\mu^2}{2} + \\ \lambda_p d_c(\gamma)(n_p(\gamma) - d_p(\gamma)) + \lambda_c d_p(\gamma)(n_c(\gamma) - d_c(\gamma)) \end{aligned}$$

The highest degree of the above equation belongs to second term of right hand side which illustrates that it is a polynomial of degree $n_p + n_c + 2$. So, the Lundberg equation has $n_p + n_c + 2$ number of distinct solutions. □

It is also clear from the expression of δ that if $\sigma^2 = 0$ and $\mu \neq 0$, then Lundberg equation has $n_p + n_c + 1$ number of roots however, if $\sigma^2 = 0$ and $\mu = 0$, then there are $n_p + n_c$ number of roots.

Lemma 5.1. *Let $\eta_i^p(\gamma) = \mathbb{E}_u(e^{\gamma V_p}|V_i^p)$ (m.g.f of overshoot) and $\eta_i^c(\gamma) = \mathbb{E}_u(e^{\gamma V_c}|V_i^c)$ (m.g.f of undershoot) and let $\zeta_i^p = \mathbb{P}_u(\tau_a < \tau_b, V_i^p)$ and $\zeta_i^c = \mathbb{P}_u(\tau_b < \tau_a, V_i^c)$, where V_p is the possible overshoot over a and V_c is the possible undershoot under b and V_i^p and V_i^c are the events of overshoot (over a) and undershoot (under b) (as stated above) respectively. Then*

$$\mathbb{E}_u[e^{\gamma R_\tau}] = e^{\gamma a}\sum_{i=0}^{n_p}\eta_i^p(\gamma)\zeta_i^p + e^{\gamma b}\sum_{i=0}^{n_c}\eta_i^c(\gamma)\zeta_i^c$$

Proof. It is clear that if the process R_t crosses the boundary (either upper or lower), then the event will happen with probability 1. Hence, if the positive jumps are of phase-type with representation $(\boldsymbol{\alpha}_p, \mathbf{T})$, then the overshoots, V_p's are also phase-type with representation $(\boldsymbol{e}_i^{'}, \mathbf{T}_p)$, where $\boldsymbol{e}_i$ is the column vector with 1 in the i^{th} position and all other entries are 0. Similarly, the undershoots are of phase-type as well with representation $(\boldsymbol{e}_i^{'}, \mathbf{T}_c)$.
If the process crosses the upper boundary a by a jump, then $R_{\tau=\tau_a} = a + V_p$. Similarly, if the process crosses the lower boundary b by a jump , then $R_{\tau=\tau_b} = b - V_c$. Therefore, the term $e^{\gamma R_\tau}$ (the right most term in modified Kella-Whitt martingale (15)) can be evaluated in the following way:
$\mathbb{E}_u[e^{\gamma R_\tau}] = \sum_{i=0}^{n_p}\mathbb{E}_u[e^{\gamma(a+V_p)}; \tau = \tau_a, V_i^p] + \sum_{i=0}^{n_c}\mathbb{E}_u[e^{\gamma(b-V_c)}; \tau = \tau_b, V_i^c]$
$= e^{\gamma a}\sum_{i=0}^{n_p}\mathbb{E}_u[e^{\gamma V_p}; \tau_a < \tau_b, V_i^p] + e^{\gamma b}\sum_{i=0}^{n_c}\mathbb{E}_u[e^{\gamma V_c}; \tau_a > \tau_b, V_i^c]$
Now, using conditional probability ($\mathbb{P}(AB) = \mathbb{P}(A|B)\mathbb{P}(B)$) the above expression can be represented as follows: (as event V_i^p occurred indicates $\tau_a < \tau_b$ already happened). Therefore, we can avoid writing of $\tau_a < \tau_b$ term inside of $\mathbb{P}_u(\cdot)$ and $\mathbb{E}_u(\cdot)$
$= e^{\gamma a}\sum_{i=0}^{n_p}\mathbb{E}_u[e^{\gamma V^+}|V_i^p]\cdot\mathbb{P}_u(V_i^p) + e^{\gamma b}\sum_{i=0}^{n_c}\mathbb{E}_u[e^{\gamma V^-}|V_i^c]\cdot\mathbb{P}_u(V_i^c)$
Additionally, $\mathbb{E}_u[e^{\gamma V_p}|V_i^p]$ represents the moment generating function of the overshoot V_p i.e. $\mathbb{E}_u[e^{\gamma V_p}|V_i^p] = \boldsymbol{e}_i^{'}(-\gamma\mathbf{I} - \mathbf{T}_p)^{-1})\mathbf{t}_p = \eta_i^p(\gamma)$. Similarly, $\mathbb{E}_u[e^{\gamma V_c}|V_i^c] = \boldsymbol{e}_i^{'}(-\gamma\mathbf{I} - \mathbf{T}_c)^{-1})\mathbf{t}_c = \eta_i^c(\gamma)$. Moreover, $\mathbb{P}_u(V_i^p)$ is the probability of the event V_i^p hence $\mathbb{P}_u(V_i^p) = \mathbb{P}_u(\tau_a < \tau_b, V_i^p) = \zeta_i^p$. Similar argument is valid for undershoot event. Hence we obtain
$\mathbb{E}_u[e^{\gamma R_\tau}] = e^{\gamma a}\sum_{i=0}^{n_p}\eta_i^p(\gamma)\zeta_i^p + e^{\gamma b}\sum_{i=0}^{n_c}\eta_i^c(\gamma)\zeta_i^c$ □

Theorem 5.2. *Assume that there exist $n = n_p + n_c + 2$ distinct complex numbers γ_i such that $\mathcal{K}(\gamma_i) = 0, i = 1, 2, \cdots, n$. Let $\eta_0^p(\gamma) = \eta_0^c(\gamma) = 1$ and $\eta_i^p(\gamma) = \mathbb{E}_u(e^{\gamma V_p}|V_i^p)$ and $\eta_i^c(\gamma) = \mathbb{E}_u(e^{\gamma V_c}|V_i^c)$ and let the solutions of the n linear equations*

$$e^{\gamma_i u} = e^{\gamma a}\sum_{i=0}^{n_p}\eta_i^p(\gamma)\zeta_i^p + e^{\gamma b}\sum_{i=0}^{n_c}\eta_i^c(\gamma)\zeta_i^c \tag{18}$$

are $\zeta_1^p, \cdots, \zeta_{n_p}^p$, $\zeta_1^c, \cdots, \zeta_{n_c}^c$, ζ_0^p, ζ_0^c.

Then

$$\mathbb{P}_u[\tau_a < \tau_b, V_i^p] = \sum_{i=0}^{n_p} \zeta_i^p \tag{19a}$$

$$\mathbb{P}_u[\tau_b < \tau_a, V_i^c] = \sum_{i=0}^{n_c} \zeta_i^c \tag{19b}$$

Proof. The proof is trivial. From (15) we see that $K_0 = 0$. According to Doobs optional stopping theorem we have $\mathbb{E}K_\tau = \mathbb{E}K_0$. Applying this on Kella-Whitt martingale given in (15) we obtain

$$\mathbb{E}_u K_0 = \mathcal{K}(\gamma) \int_0^\tau e^{\gamma R_s} ds + e^{\gamma u} - \mathbb{E}_u[e^{\gamma R_\tau}] \tag{20}$$

However, we see that for all $\gamma \in \mathbb{C}$, $\mathbb{E}_u \int \{\cdot\}$ is an analytic function as

$$0 \leq \mathbb{E}_u \int_0^\tau e^{\gamma Z_s} ds \leq \mathbb{E}_u \tau e^{\gamma(a+|b|)}$$

for any $\gamma \in \mathbb{C}$.
Now according to corollary (5.1) the Lundberg equation holds i.e. there exist n complex number $\gamma_i, i = 1, 2, \cdot, n$ such tat $\mathcal{K}(\gamma) = 0$ which eliminates the first term of the right side. Thus

$$\mathbb{E}_u K_0 = e^{\gamma u} - \mathbb{E}_u[e^{\gamma R_\tau}]$$

However, $\mathbb{E}_u K_0 = K_0 = 0$ then the above expression simplifies to $e^{\gamma u} = \mathbb{E}_u[e^{\gamma R_\tau}]$. Using lemma (5.1) above equation can be expressed as follows:

$$e^{\gamma u} = e^{\gamma a} \sum_{i=0}^{n_p} \eta_i^p(\gamma) \zeta_i^p + e^{\gamma b} \sum_{i=0}^{n_c} \eta_i^c(\gamma) \zeta_i^c \tag{21}$$

Which implies that for $\gamma = \gamma_i, i = 1, 2, \cdots, n$ there are n linear equations. Additionally, ζ_i^p's and ζ_i^c's are the only unknown in equation (21). Let's define $\zeta_i^p = \mathbb{E}_u[\mathbb{1}_{\tau_a < \tau_b}, V_i^p]$ and $\zeta_i^c = \mathbb{E}_u[\mathbb{1}_{\tau_b < \tau_a}, V_i^c]$, then it is clear that

$$\begin{aligned}\mathbb{P}_u[\tau_a < \tau_b] &= \mathbb{E}_u[\mathbb{1}_{\tau_a < \tau_b}, V_i^p] \\ &= \mathbb{E}_u[\mathbb{1}_{\tau_a < \tau_b}, V_0^p] + \mathbb{E}_u[\mathbb{1}_{\tau_a < \tau_b}, V_1^p] + \cdots + \mathbb{E}_u[\mathbb{1}_{\tau_a < \tau_b}, V_{n_p}^p] \\ &= \sum_{i=0}^{n_p} \zeta_i^p\end{aligned}$$

Similarly, $\mathbb{P}_u(\tau_b < \tau_a) = \sum_{i=0}^{n_c} \zeta_i^c$.
However, taking $b = 0$ it is possible to obtain the probability of up-crossing before ruin and vice-versa. □

6 Example

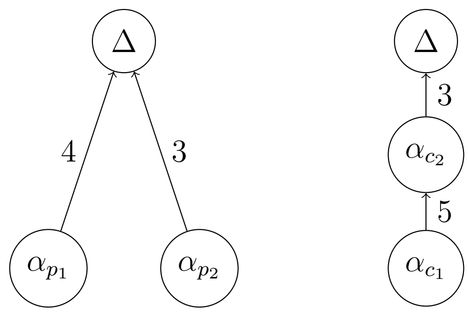
Let the premiums, p_i are of phase-type with representation $(\alpha_p, \mathbf{T}_p)$, where

$$\mathbf{T}_p = \begin{pmatrix} -4 & 0 \\ 0 & -3 \end{pmatrix}, \quad \alpha_p = \begin{pmatrix} \frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

And the claims, c_i are also phase-type with representation

$$\mathbf{T}_c = \begin{pmatrix} -5 & 5 \\ 0 & -3 \end{pmatrix}, \quad \alpha_c = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Graphical representation of the premium and claim sizes distributions are as follows:



According to equation (4) density of premiums is

$$f_p(x) = \frac{15}{7}e^{-3x} + \frac{8}{7}e^{-4x} \tag{22}$$

and density of claims is

$$f_c(x) = \frac{21}{4}e^{-3x} - \frac{15}{4}e^{-5x} \tag{23}$$

For simplicity, let rate of the positive jumps, $\lambda_p = 3$ and rate of the negative jumps, $\lambda_c = 2$. Then according to equation (9) Lévy exponent of our premium process is

$$\mathcal{K}_p(\gamma) = \lambda_p \left(\alpha_p \left(-\gamma \mathbf{I} - \mathbf{T}_p\right)^{-1} \mathbf{t}_p - 1\right) = 3\left(\frac{26\gamma - 7\gamma^2}{7(4-\gamma)(3-\gamma)}\right)$$

and Lévy exponent of our claim process is

$$\mathcal{K}_c(\gamma) = 2\left(\frac{-2\gamma^2 - 13\gamma}{2(5+\gamma)(3+\gamma)}\right)$$

Moreover, considering $\mu = 0$ (drift of B.M.) and $\sigma^2 = 1$ (variance of B.M.), we obtain Lévy exponent our risk process is

$$\mathcal{K}(\gamma) = \frac{\gamma^2}{2} + \frac{78\gamma - 21\gamma^2}{7(4-\gamma)(3-\gamma)} - \frac{2\gamma^2 + 13\gamma}{(5+\gamma)(3+\gamma)} \tag{24}$$

Therefore, according to corollary (5.1) there are 6 complex numbers satisfying $\mathcal{K}(\gamma_i) = 0$ which are $\gamma_1 = 0,\quad \gamma_2 = -0.0551665,\quad \gamma_3 = 3.59869,\quad \gamma_4 = 4.86516,\quad \gamma_5 = -4.70434 - 0.97082i$ and $\gamma_6 = -4.70434 + 0.97082i$.

Hence, according to (18), we have the following system of 6 linear equations

$$\begin{aligned}
e^{\gamma_1 u} &= e^{\gamma_1 a}\{\eta_0^p(\gamma_1)\zeta_0^p + \eta_1^p(\gamma_1)\zeta_1^p + \eta_2^p(\gamma_1)\zeta_2^p\} + e^{\gamma_1 b}\{\eta_0^c(\gamma_1)\zeta_0^c + \eta_1^c(\gamma_1)\zeta_1^c + \eta_2^c(\gamma_1)\zeta_2^c\}\\
e^{\gamma_2 u} &= e^{\gamma_2 a}\{\eta_0^p(\gamma_2)\zeta_0^p + \eta_1^p(\gamma_2)\zeta_1^p + \eta_2^p(\gamma_2)\zeta_2^p\} + e^{\gamma_2 b}\{\eta_0^c(\gamma_2)\zeta_0^c + \eta_1^c(\gamma_2)\zeta_1^c + \eta_2^c(\gamma_2)\zeta_2^c\}\\
e^{\gamma_3 u} &= e^{\gamma_3 a}\{\eta_0^p(\gamma_3)\zeta_0^p + \eta_1^p(\gamma_3)\zeta_1^p + \eta_2^p(\gamma_3)\zeta_2^p\} + e^{\gamma_3 b}\{\eta_0^c(\gamma_3)\zeta_0^c + \eta_1^c(\gamma_3)\zeta_1^c + \eta_2^c(\gamma_3)\zeta_2^c\}\\
e^{\gamma_4 u} &= e^{\gamma_4 a}\{\eta_0^p(\gamma_4)\zeta_0^p + \eta_1^p(\gamma_4)\zeta_1^p + \eta_2^p(\gamma_4)\zeta_2^p\} + e^{\gamma_4 b}\{\eta_0^c(\gamma_4)\zeta_0^c + \eta_1^c(\gamma_4)\zeta_1^c + \eta_2^c(\gamma_4)\zeta_2^c\}\\
e^{\gamma_5 u} &= e^{\gamma_5 a}\{\eta_0^p(\gamma_5)\zeta_0^p + \eta_1^p(\gamma_5)\zeta_1^p + \eta_2^p(\gamma_5)\zeta_2^p\} + e^{\gamma_5 b}\{\eta_0^c(\gamma_5)\zeta_0^c + \eta_1^c(\gamma_5)\zeta_1^c + \eta_2^c(\gamma_5)\zeta_2^c\}\\
e^{\gamma_6 u} &= e^{\gamma_6 a}\{\eta_0^p(\gamma_6)\zeta_0^p + \eta_1^p(\gamma_6)\zeta_1^p + \eta_2^p(\gamma_6)\zeta_2^p\} + e^{\gamma_6 b}\{\eta_0^c(\gamma_6)\zeta_0^c + \eta_1^c(\gamma_6)\zeta_1^c + \eta_2^c(\gamma_6)\zeta_2^c\}
\end{aligned}$$

Additionally, according to the statement of theorem (5.2), we have

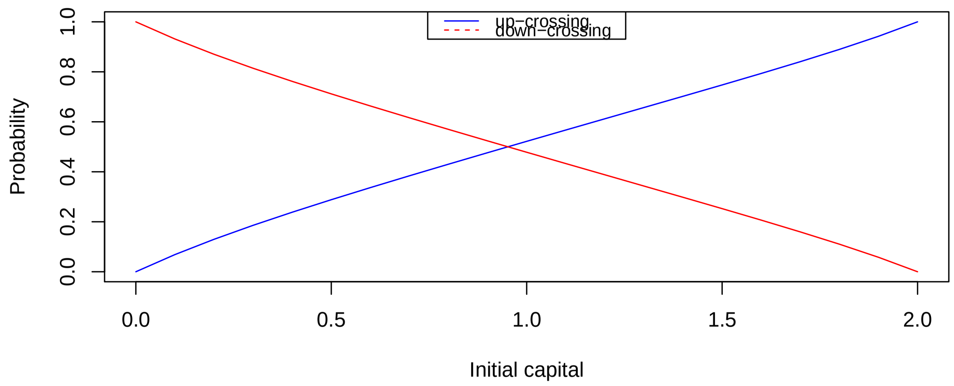
$\eta_0^p(\gamma) = \eta_0^c(\gamma) = 1,\quad \eta_1^p(\gamma) = \dfrac{4}{4-\gamma},\quad \eta_2^p(\gamma) = \dfrac{3}{3-\gamma},\quad \eta_1^c(\gamma) = \dfrac{15}{(\gamma+5)(\gamma+3)}$, and $\eta_2^c(\gamma) = \dfrac{3}{(\gamma+3)}$.

By substituting the values of γ_i's, η_i^p's, η_i^c's and by using $a = 2$, $b = 0$ in the above system of linear equations, we obtain their following form

$$
\begin{aligned}
e^{u} &= e^{2}\left(\zeta_0^p + \frac{4}{4-1}\zeta_1^p + \frac{3}{3-1}\zeta_2^p\right) + \zeta_0^c + \frac{15}{(5+1)(3+1)}\zeta_1^c + \frac{3}{3+1}\zeta_2^c \\
e^{-1.02415u} &= e^{-1.02415*2}\left(\zeta_0^p + \frac{4}{4+1.02415}\zeta_1^p + \frac{3}{3+1.02415}\zeta_2^p\right) + \zeta_0^c + \frac{15}{(5-1.02415)(3-1.02415)}\zeta_1^c + \frac{3}{3-1.02415}\zeta_2^c \\
e^{3.62701u} &= e^{3.62701*2}\left(\zeta_0^p + \frac{4}{4-3.62701}\zeta_1^p + \frac{3}{3-3.62701}\zeta_2^p\right) + \zeta_0^c + \frac{15}{(5+3.62701)(3+3.62701)}\zeta_1^c + \frac{3}{3+3.62701}\zeta_2^c \\
e^{4.98128u} &= e^{4.98128*2}\left(\zeta_0^p + \frac{4}{4-4.98128}\zeta_1^p + \frac{3}{3-4.98128}\zeta_2^p\right) + \zeta_0^c + \frac{15}{(5+4.98128)(3+4.98128)}\zeta_1^c + \frac{3}{3+4.98128}\zeta_2^c \\
e^{(-4.79207-1.0214i)u} &= e^{(-4.79207-1.0214i)*2}\left(\zeta_0^p + \frac{4}{4-(-4.79207-1.0214i)}\zeta_1^p + \frac{3}{3-(-4.79207-1.0214i)}\zeta_2^p\right) + \\
&\quad \zeta_0^c + \frac{15}{(5+(-4.79207-1.0214i))(3+(-4.79207-1.0214i))}\zeta_1^c + \frac{3}{3+(-4.79207-1.0214i)}\zeta_2^c \\
e^{(-4.79207+1.0214i)u} &= e^{(-4.79207+1.0214i)*2}\left(\zeta_0^p + \frac{4}{4-(-4.79207+1.0214i)}\zeta_1^p + \frac{3}{3-(-4.79207+1.0214i)}\zeta_2^p\right) + \\
&\quad \zeta_0^c + \frac{15}{(5+(-4.79207+1.0214i))(3+(-4.79207+1.0214i))}\zeta_1^c + \frac{3}{3+(-4.79207+1.0214i)}\zeta_2^c
\end{aligned}
$$

Now taking different values of u and using formula (19), we obtained probabilities of up-crossing before ruin. The tabular and graphical presentations are as follows:

u	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\mathbb{P}_u(\tau_a < \tau_0)$	0.000	0.032	0.061	0.087	0.113	0.140	0.166	0.194	0.224	0.256
0.291	0.329	0.370	0.416	0.467	0.524	0.590	0.665	0.755	0.864	1.000



7 Conclusion

In this paper, we have assumed that the reserve of an insurer follows Lévy process. However, if the Lévy process have both sided jumps, where both of the jumps are of phase-type, then using numericl example, we see that the theorem for probability of up-crossing before ruin and vice-versa given in [5] works perfectly..

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