Optimal Consumption and Annuity Equivalent Wealth with Mortality Model Uncertainty

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Optimal Consumption and Annuity Equivalent Wealth with Mortality Model Uncertainty

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Executive Summary

The classic Yaari lifecycle model (LCM) lies at the very heart of much modern retirement research, particularly the economic understanding of annuity demand. The LCM predicts a high annuity demand among individuals facing retirement, yet it is rarely the case in reality. Such a disconnect between economic theory and practice is known as the annuity puzzle, spurring intensified research attempting to demystify its economic and psychological underpinnings.

In this paper, we aim to understand the cause of low annuity demand through the angle of mortality model uncertainty. To this end, we advance Yaari's LCM via incorporating a mortality model uncertainty analysis (a.k.a. perturbation analysis) and obtain the optimal robust consumption policies. Under an uncertain mortality environment, we examine the annuity equivalent wealth (AEW) and discover that investors may understate the value of an annuity if mortality model[1] uncertainty is ignored. Based on a realistic choice of parameters, the following findings are obtained:

- The worst-case mortality scenario in the perturbation analysis corresponds to an improved mortality trajectory, implying that a retiree's aversion against mortality uncertainty is translated into the fear of longevity risk in retirement planning.

- The worst-case perturbed mortality model is a parallel shift (also known as the proportional hazard distortion) of the best-estimated reference mortality curve.

- Even under the presence of mortality model uncertainty, annuitization can still induce a noticeable increment in utility.

- The optimal annuity payout pathway increases over time as mortality rate grows and more uncertainty about the future is resolved during the later stage of retirement.

- The presence of mortality ambiguity aversion increases the value of annuity equivalent wealth.

The last finding outlined above reveals that if mortality model uncertainty is disregarded, then the actual economic welfare gained by annuitization may be undervalued, causing fewer retirees to purchase annuities. Our study acknowledges that retirees should not place a full conviction on a specific mortality assumption. Otherwise, the longevity risk inherent in retirement planning will be underestimated. Educating investors to recognize the uncertainty around future mortality pathways may be one of the possible ways to resolve the enduring issue of low annuity demand in the present retirement market.

[1] A model is a simplified mathematical description of some real-world phenomena. A mortality model can be as simple as life expectancies or as complex as stochastic mortality curves. Admittedly, some sort of mortality model is likely to be involved in a retiree's decision process.

Section 1: Introduction

Owing to the growing public concern on retirement funding inadequacy, retirement planning has become a very active research area in the actuarial community during recent decades. A constant focus has been placed on the study about how retirees should wisely draw down their retirement nest eggs in order to receive adequate protection against the risks of outliving their retirement savings. Toward this aim, actuarial researchers resort to the rational economic theory for obtaining the optimal blueprint of retirees' saving and consumption behaviors. Originally postulated in Fisher (1930) under the assumption of deterministic time horizon and then refined by Yaari (1965) to a stochastic lifetime, the lifecycle model (LCM) of consumption has evolved as the building block in much modern retirement research. Namely, Yaari (1965) derived the optimal consumption rule for a utility-maximizing retiree facing a stochastic time of death under an additive utility function. Yarri (1965) found that if there is no bequest motive, then the rational investor should convert all the savings into an actuarially fair annuity upon retirement. Later, rigorous analysis by Davidoff, et al. (2005) shows that many model assumptions in Yaari (1965) can be relaxed, while the original conclusion on full annuitization still remains true.

Though the economic theory predicts a high annuity demand, this is rarely the case in reality. Very few consumers facing retirement choose to annuitize a substantial portion of their retirement savings (Benartzi et al., 2011). This disparity between theory and the actual consumers' behavior, commonly referred to as the annuity puzzle, has spurred intensified research attempting to demystify the economic and psychological underpinnings. Several explanations for the annuity puzzle have been proposed, including low retirement savings amongst the population (Dushi and Webb, 2004), less flexibility to control spending (Pang and Warshawsky, 2010; Peijnenburg et al., 2017), the presence of bequest motive (Lockwood, 2012), incomplete annuity market (Horneff et al., 2008; Koijen et al., 2011), unfair annuity pricing (Mitchell et al., 1999), and default risk of the annuity providers (Agnew et al., 2008), to name but only a few. It is fair to state that none of these explanations alone can fully account for low annuity demand in the market, but the aforementioned studies together essentially help us to better understand the issue from different angles.

This paper bears another effort to unravel the annuity puzzle via the angle of mortality model uncertainty. With all the other complexities involved in retirement planning, the subjective assessment of a retiree's future mortality pathway plays a decisive role in the decision process. Deviations from the mortality prediction model may pose a substantial influence on the lifespan discounted utility, thus turning the initially optimal strategy to be sub-optimal. Nevertheless, modeling individual mortality is notoriously hard from a statistical standpoint. Different from the objective mortality model which can be estimated from the population data, the micro-structure of the subjective mortality is extremely complicated and is closely related to the retiree's occupation, wealth, lifestyle, and other socioeconomic determinants (Hurd and McGarry, 1995, 2002). To develop an effective retirement strategy, the subjective mortality model should be "best-estimated'' using available data, while one must be also mindful of the model risk associated with the best-estimated model.

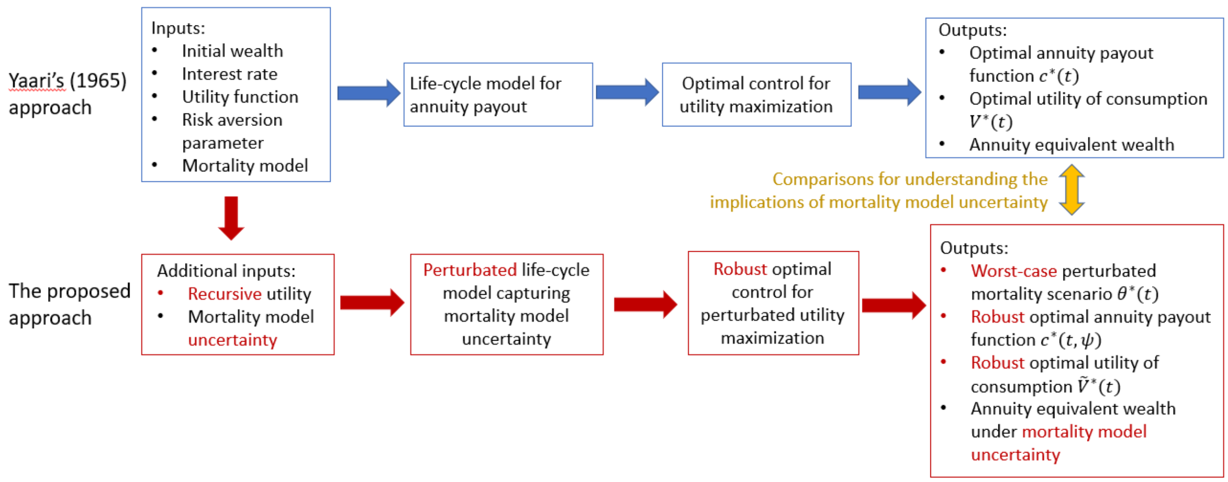
In this paper, we treat the uncertainty surrounding the subjective mortality model as a robust control problem. That is, in addition to the best-estimated reference mortality model, we should consider an alternative set of statistically similar mortality models, among which we solve the retirement planning problem based on the so-called endogenously determined worst-case mortality scenario. Consequently, the consumption strategy obtained in our study will remain desirable even when the best-estimated mortality model performs inadequately. As a side note, in behavioral economics, an investor's fear of the uncertainty in the estimated probability distributions of future outcomes, is referred to as ambiguity aversion. In the context of this current paper, mortality ambiguity aversion represents a retiree's concern about the mortality model uncertainty. If a retiree has no mortality ambiguity aversion, that means the retiree will choose to ignore the uncertainty surrounding the mortality model and fully trust the best-estimated mortality curve.

The rest of this paper is organized as follows. Section 2 provides a high-level summary of the technical approach and findings as well as numerical illustrations. After revisiting Yaari's LCM in Section 3, we set up an LCM with mortality model uncertainty and recursive utility in Section 4. Section 5 derives the optimal consumption strategies and discusses the economic implications of mortality model uncertainty. Section 6 states our conclusions. In order to facilitate the reading, all technical proofs are relegated to Appendix A. Throughout, we consider a probability space $(\Omega, F, \mathbb{P})$ satisfying the usual conditions, in which $\mathbb{P}$ is a reference probability measure. In this paper, we focus on the impact of mortality model uncertainty on the retiree's optimal demand for a life annuity, while not addressing investment model uncertainty.

Section 2: Summary of the Approach and Findings

2.1 THE TECHNICAL APPROACH

Arguably, the annuity puzzle itself is a mathematical economics problem. In order to demystify the problem, highly sophisticated technical tools are inevitably involved. At a high level, the following flowchart summarizes the technical approach adopted in our paper.



Motivated by the original study of LCM in Yaari (1965), two market conditions will be considered. They are namely the complete annuity (CA) market, which refers to the availability of a complete set of life-only annuities[2] at actuarially fair prices, and the complete bond (CB) market, wherein pure discount bonds are available for any maturities, but annuities are absent. For the reader's convenience, below is a table summarizing the notation system.

[2] We openly admit that this paper only focuses on life-only annuities due to their simple structures. Like most of economics research which require a significant level of abstraction of the real world, we hope that our results derived based on this simplest annuity product can be extended to another more realistic yet complicated scenario. Follow-up research is still needed to verify this conjecture.

Notation	Description
${}_tp_y$	Probability of an individual aged at y survives more than t years
λ_x	Mortality rate (i.e., force of mortality) at age x
r	Risk-free discount rate (for discounting cashflows)
ρ	Subjective discount rate (for discounting utility)
γ	Risk aversion parameter (the higher the value of γ, the more averse a retiree becomes against consumption fluctuations due to the random states of the future)
ϕ	Elasticity of intertemporal substitution (EIS) of consumption (the higher the value of ϕ, the less averse a retiree becomes against consumption fluctuations over time in a deterministic world)
ψ	Model ambiguity aversion parameter (the higher the value of ψ, the more averse a retiree becomes against uncertainties surrounding the "best estimated" model)
$\theta(t)$ and $\theta^*(t)$	Mortality perturbation function such that the perturbated mortality curve is $\theta_t \times \lambda_{x+t}$, and the worst-case perturbation function is denoted by the superscript "*"
$X_A(t;\psi)$ and $X_B(t;\psi)$; $X_A^*(t;\psi)$ and $X_B^*(t;\psi)$	Wealth processes at time t under the complete annuity market condition (A) and complete bond market (B), respectively, and the optimal counterparts are denoted by the superscript "*"
$V_A(t;\psi)$ and $V_B(t;\psi)$; $V_A^*(t;\psi)$ and $V_B^*(t;\psi)$	Discounted utilities at time t under the complete annuity market condition (A) and complete bond market (B), respectively, and the optimal counterparts are denoted by the superscript "*"
$c_A(t;\psi)$ and $c_B(t;\psi)$; $c_A^*(t;\psi)$ and $c_B^*(t;\psi)$	Annuity payout functions at time t under the complete annuity market condition (A) and complete bond market (B), respectively, and the optimal counterparts are denoted by the superscript "*"
$\pi_A(t;\psi)$ and $\pi_B(t;\psi)$; $\pi_A^*(t;\psi)$ and $\pi_B^*(t;\psi)$	Consumption-to-wealth ratios at time t under the complete annuity market condition (A) and complete bond market (B), respectively, and the optimal counterparts are denoted by the superscript "*"
AEW	Annuity equivalent wealth, which quantifies the utility increment gained by annuitization

2.2 SUMMARY OF THE CONTRIBUTIONS AND FINDINGS

Here is a summary of our paper's technical results and economic insights, which we believe are new to the literature.

1. In terms of the technical contributions, we extend Yaari's LCM in two innovative aspects. First, we integrate a perturbation analysis into the study of Yaari's LCM and obtain the optimal consumption policies that are robust to the uncertainty occurred to the best-estimated mortality model. Inspired by the original study in Yaari (1965), both the complete annuity market and the complete bond market conditions are considered, and under an endogenous worst-case mortality scenario, we calculate the annuity equivalent wealth (AEW) which quantifies the utility increment gained by annuitization. Second, we generalize the additive utility considered in Yaari's LCM to the more general Epstein-Zin recursive utility (Duffie and Epstein, 1992). Particularly, recent studies of Yaari's LCM often assume the constant relative risk aversion (CRRA) preferences. It is known that the CRRA utility restricts the risk aversion parameter to be the reciprocal of the elasticity of intertemporal substitution parameter. See Remark 1 for details. However, these two parameters characterize very distinct features of a retiree's risk preferences. The adoption of recursive utility allows us to distinguish the coefficient of relative risk aversion from the EIS in consumption. It is interesting to study what the roles of these two different risk preference parameters play in quantifying the retiree's perception about mortality model uncertainty, the associated optimal consumption policies, and AEW. For more details, we refer the reader to Sections 3 and 4.

2. Capitalizing on the aforementioned extensions of Yaari's LCM, we find that compared with the reference mortality model, the worst-case mortality scenario can be a deteriorated or improved mortality pathway, depending on the value of EIS. If the retiree's EIS ϕ is smaller than one, which is the common case as shown in the empirical study by Yogo (2004), then $\theta^* < 1$ and so the worst-case mortality scenario corresponds to an improved mortality trajectory, implying that the retiree is more concerned about the longevity risk throughout the retirement planning phase. This is a rather non-trivial yet appealing finding, which shows that the recent research focus on longevity risk is even meaningful from the mortality model uncertainty standpoint. For more details, we refer the reader to Theorem 1 in Section 5.1 and the discussions after it.

3. Moreover, we find that the worst-case perturbed mortality model corresponds to a parallel shift (also known as the proportional hazard distortion; see, Wang, 1996) of the best-estimated reference mortality curve. This type of distortion is often adopted to examine the sensitivity of mortality assumption in retirement research, mainly due to its inherent simplicity. Our study shows the choice of parallel shock is indeed sufficiently conservative for covering the worst-case mortality scenario, so far at least as the LCM is concerned. For more details, we refer the reader to Theorem 1 and the discussions in Section 5.1.

4. Realizing mortality model uncertainty, it is discovered that the optimal annuity payout pathway may present an asymmetric U-shaped pattern over time. Based on reasonable values of parameters, the optimal annuity payout function decreases over time at the beginning of the retirement phase when the force of mortality is still relatively low. This is because the expected remaining lifetime of the retiree is still long. A substantial number of uncertainties still remain over the outstanding retirement life, so the retiree needs to save more wealth for the future. However, as the retiree gets older, the mortality rate grows and more uncertainty about the future is resolved for the remaining life, the optimal annuity payout function will essentially become increasing. Conventional annuity products may be redesigned to incorporate such a payment pattern to be more attractive to retirees. While annuity products having this type of payout pattern are not common (or may not even exist) in the insurance market, it will be interesting to investigate their marketability in follow-up research. The optimal drawdown non-annuity strategies may present an asymmetric U-shaped pattern over time. For more details, we refer the reader to Proposition 4 and the discussions in Section 5.1.

5. Compared with the original LCM studied in Yaari (1965), the presence of mortality model uncertainty decreases the optimal annuity payout/consumption amount at the beginning of the retirement phase but increases the amount during the latter stage. The optimal consumption-to-wealth ratio decreases because of mortality model uncertainty. This is because, as argued in Point 1, a retiree's aversion against mortality uncertainty is translated into the fear of longevity risk in retirement planning. For more details, we refer the reader to Propositions 6 - 7 and the discussions in Section 5.2.

6. Even under the presence of mortality model uncertainty, annuitization can still induce a noticeable utility increment and consumption increment. For more details, we refer the reader to the numerical example and Proposition 4.

7. Under a realistic parameter setting, we show rigorously that the presence of mortality ambiguity aversion increases the value of AEW. This suggests a brand-new angle to understand the enduring economic puzzle on low annuity demand. Namely, if mortality model uncertainty is ignored, then the actual economic welfare gained by annuitization may be undervalued, causing fewer retirees to purchase annuities. Our study acknowledges that retirees should not place a full conviction on a specific mortality assumption. Otherwise, the longevity risk inherent in retirement planning will be underestimated. Educating investors to recognize the uncertainty around future mortality pathways may be one of the possible ways to resolve the issue of low annuity demand in the present retirement market. For more details, we refer the reader to Proposition 8 and the discussions in Section 5.2.

2.3 A NUMERICAL SUMMARY OF THE RESULTS

In this subsection, we present a numerical example to illustrate the findings of our paper, which were summarized in the previous section. For more details of our model, technical results, and economic arguments, we refer the reader to Sections 3 – 5. Suppose that the rational retiree of interest is now aged 65 and endowed with a retirement saving of \$100 (thousand). We estimate the baseline mortality curve using the celebrated Gompertz law (Gompertz, 1825; also see Milevsky, 2020 for a recent development):

$$\lambda_x^{\mathrm{GM}} = w_1\, exp(w_2 x), \quad x, w_1, w_2 > 0. \tag{1}$$

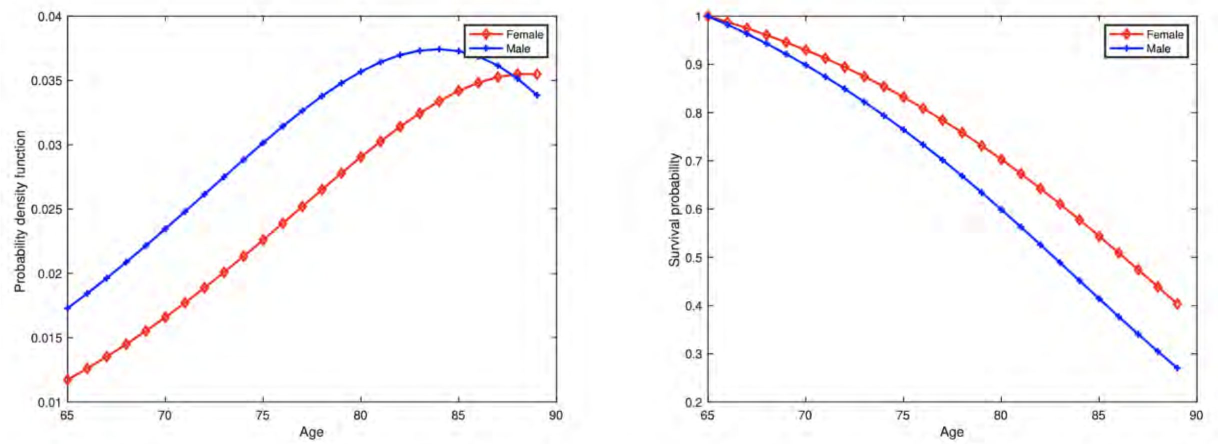
We fit the mortality model (1) into the 2015 - 2019 U.S. mortality table extracted from the Human Mortality Database[3]. The parameters are estimated to be $\widehat{w}_1 = 5.01 \times 10^{-5}$ and $\widehat{w}_2 = 8.39 \times 10^{-2}$ for female, and $\widehat{w}_1 = 8.10 \times 10^{-5}$ and $\widehat{w}_2 = 8.25 \times 10^{-2}$ for male. Figure 1 depicts the probability density function as well as the survival probability associated with the fitted Gompertz mortality model. As is shown in the right panel of Figure 1, the female survival probability curve dominates that of male, implying that female is more likely to survive longer than male at any future time points.

Moreover, we set the risk-free rate to be 1.9% according to the U.S. cash rate in the 2021 Long-Term Capital Market Assumptions report published by the J.P. Morgan Asset Management. We set the subjective discount rate to be 3% which is a standard choice in the related literature. Indeed, Frederic et al. (2002) conducted a comprehensive literature review on estimated discount rates in previous studies and found a predominance of high discount rates, being well above market interest rate. The choices of the EIS coefficient ϕ and robustness parameter ψ are rather

[3] We recognize the mortality assumption used in annuity pricing is likely to be different from the one estimated from the Human Mortality Database. To address the difference, one can apply a loading factor in front of λ. However, the discrepancy or adjustment will pose no harm to our economic findings. The dependence between mortality model uncertainty and annuity demand will remain the same even under a different baseline mortality assumption. For this reason, we shall simply stick with the mortality rate estimated from the Human Mortality Database throughout the numerical study.

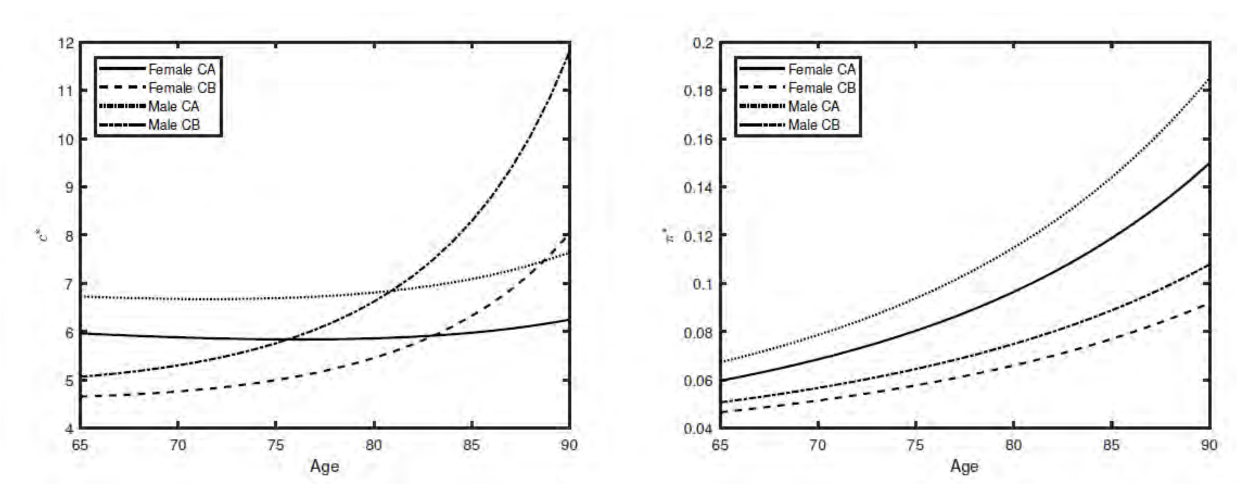
subjective, which depends heavily on the retiree's individual risk preference. Motivated by the empirical study by Yogo (2004), we set $\phi = 0.5$ and $\psi = 1$ as the baseline parameters, which will be then shocked to understand their implications on the optimal retirement strategies.

Figure 1 PROBABILITY DENSITY FUNCTION (LEFT PANEL) AND SURVIVAL PROBABILITY FUNCTION (RIGHT PANEL) OF THE RETIREE'S REMAINING LIFETIME RV τ_{65}.



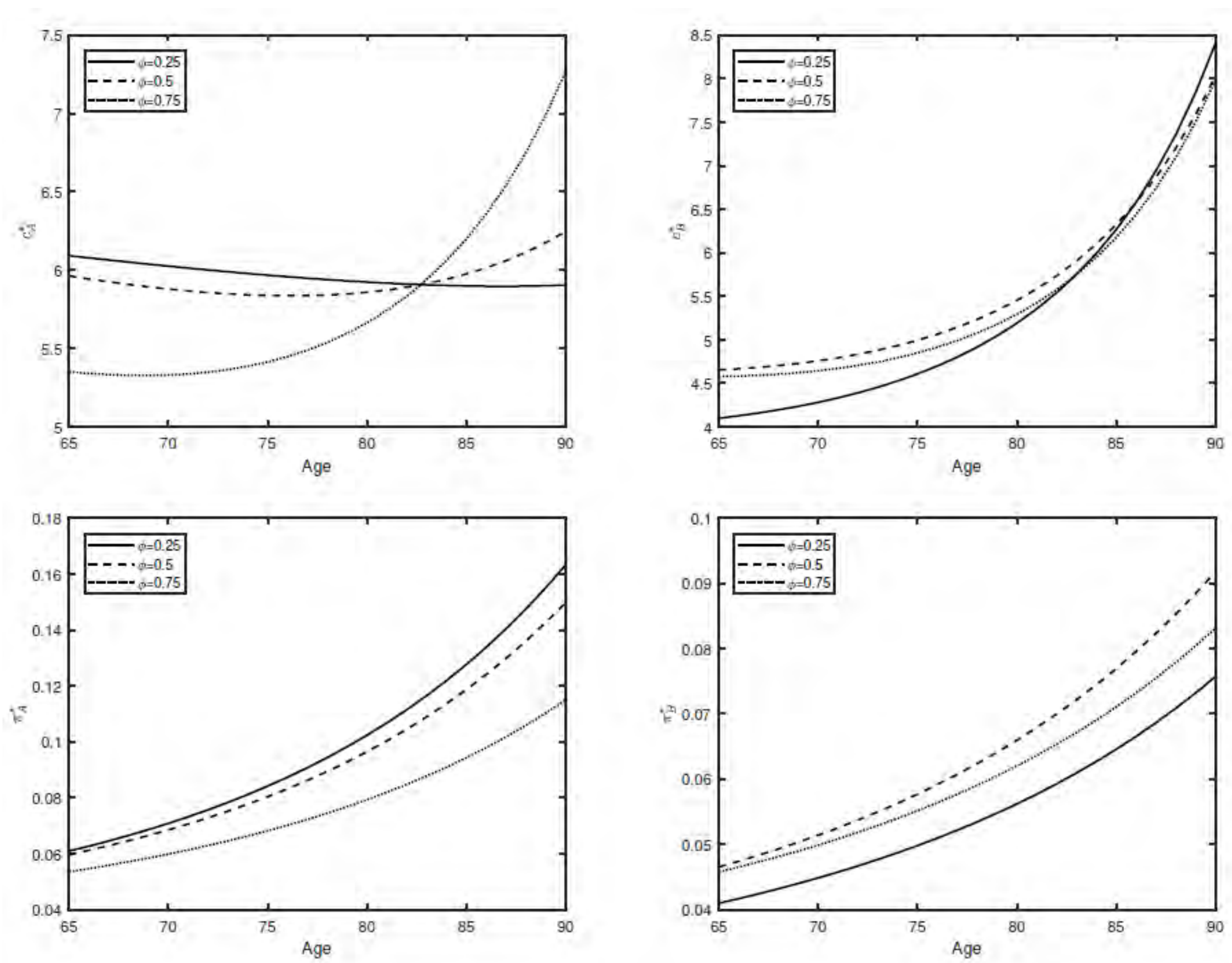
Based on the aforementioned baseline parameters, Figure 2 presents the optimal consumption pathways underlying the proposed robust LCM (also see, Theorem 1). We find that the optimal consumption amount and consumption rate for males are higher than those of females. This is intuitive because the life expectancy of females is longer than males, the female retiree will rationally lower the consumption in order to save more wealth for the future. Meanwhile, for both the female and male retirees, the optimal consumption amount under the CA market is higher than that under the CB market at the beginning of the retirement phase. This occurs because by purchasing an annuity, the retiree earns higher returns than by purchasing a bond due to the mortality credit. However, when it comes to the optimal consumption rate, it always holds that $\pi_A^* > \pi_B^*$. For more rigorous theoretical investigations, we refer the reader to Proposition 4 in Section 5.

Figure 2 THE OPTIMAL CONSUMPTION AMOUNT (LEFT PANEL) AND OPTIMAL CONSUMPTION-TO-WEALTH RATIO (RIGHT PANEL) UNDER THE BASELINE PARAMETERS.



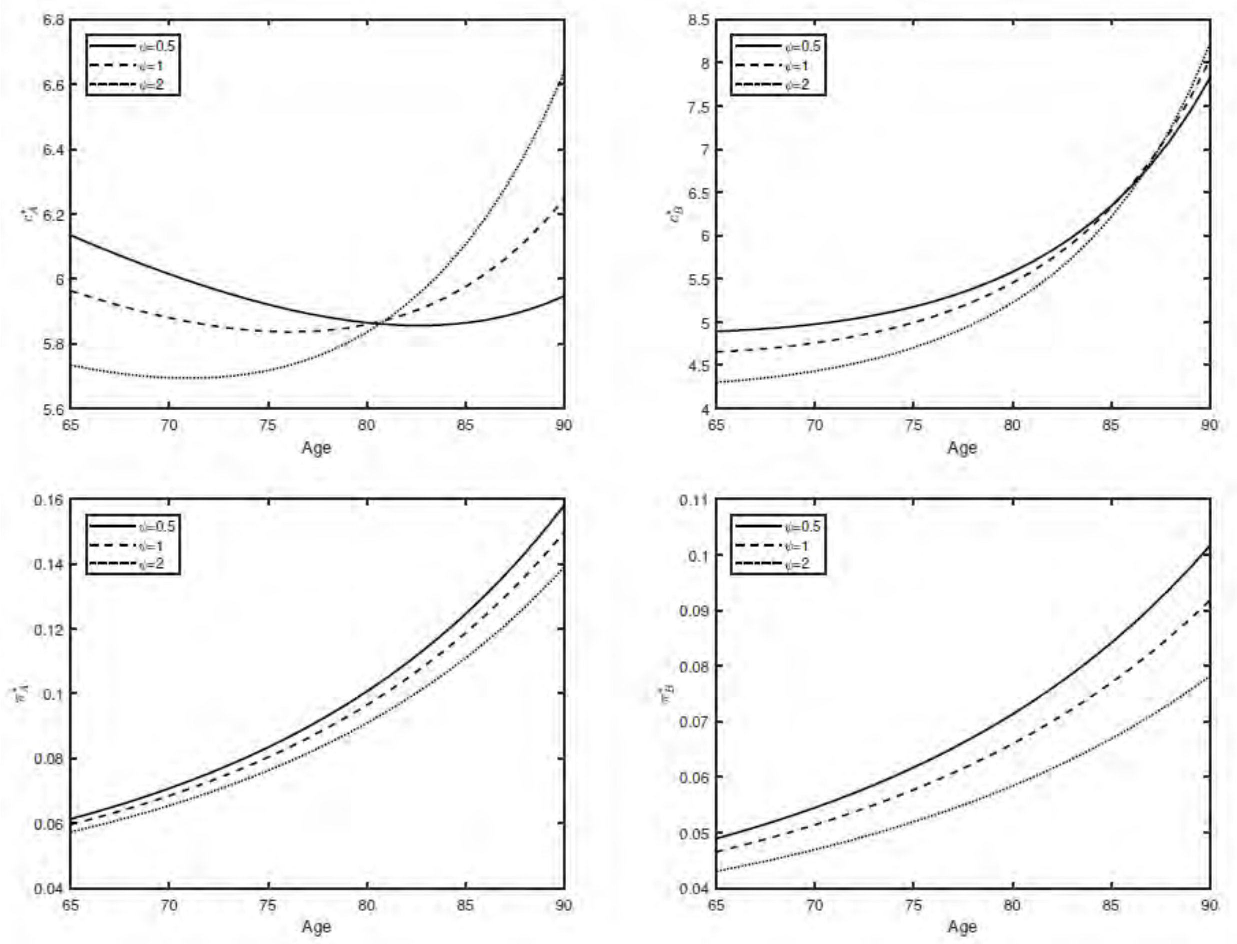
Next, we are going to study the sensitivity of the EIS parameter $\phi \in (0,1)$. As mentioned in Section 2.1, an empirical study conducted by Yogo (2004) confirmed that the EIS parameter for a typical investor falls into this range. As already seen in Figure 2 the optimal consumption rules between females and males have the same pattern, thus we focus on the female retiree only. From Figure 3, we observe that under the CA market condition, a smaller EIS leads to a flatter optimal consumption function over time, and the optimal consumption-to-wealth ratio tends to be higher. However, the optimal consumption pathways under the CB market condition do not seem to have a monotonic pattern in response to the change in the EIS coefficient. This may be caused by the fact that the EIS parameter ϕ determines not only the optimal consumption decision but also the worst-case perturbed mortality model, which complicates the impact of ϕ on the optimal robust consumption rule.

Figure 3 The optimal consumption amount (first row and optimal consumption-to-wealth ratio (second row with varying $\phi \in \{0.25,0.5,0.75\}$. The left panels recognize CA and the right are CB.



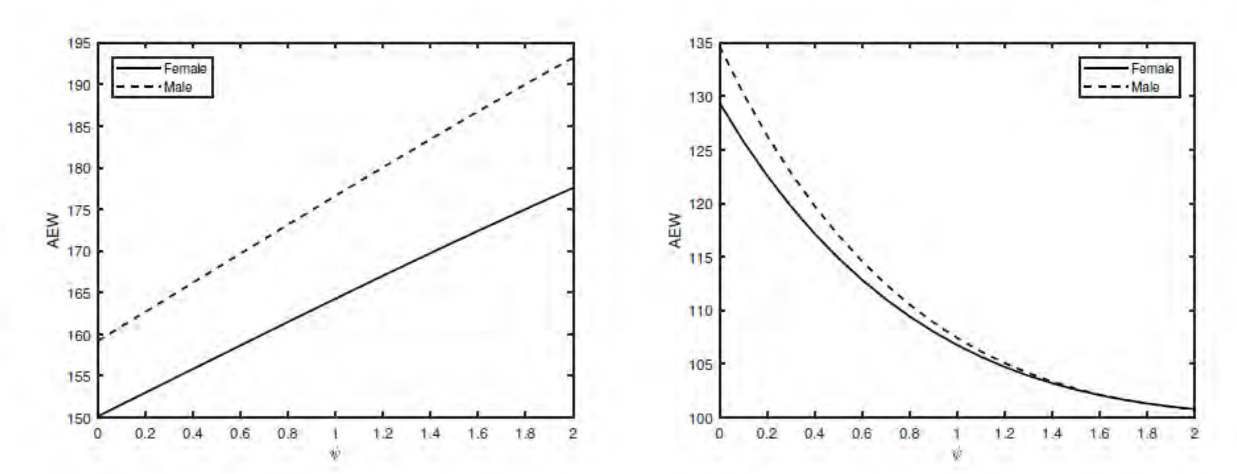
Different than the EIS coefficient, the study of the sensitivity of the robustness preference parameter $\psi > 0$ is much more predictable. According to the left panel of Figure 4, we find that a lower robustness preference ψ increases the optimal consumption amount at the beginning of the retirement phase. The pattern is intuitive because when $\phi < 1$, the longevity risk is more concerned (see Point 2 in Section 2.2), and the retiree should rationally reduce the consumption amount at the beginning of the retirement phase so as to keep more savings for the future. In another unreported case with EIS $\phi > 1$, and thus the mortality risk is of more concern, the rational retiree will choose to increase the consumption amount at the beginning of the retirement phase to make sure that a desirable amount of consumption utility can be gained before death. The right panel of Figure 4 shows that when EIS $\phi < 1$, the mortality ambiguity aversion will lower the percentage of consumption out of the present wealth at every instant, no matter whether or not an annuity is purchased. This is because if $\phi < 1$, then the worst-case perturbation function satisfies $\theta^* < 1$, which corresponds to a longevity risk scenario. Consequently, the retiree reduces the consumption rate in order to lower the risk of outliving retirement savings. Alternatively, if EIS $\phi > 1$, then the worst-case perturbed mortality curve corresponds to a mortality risk scenario. Thus, in order to maximize the lifespan discounted utility, the rational retiree will choose to increase the consumption ratio.

Figure 4 The optimal consumption amount (first row) and optimal consumption-to-wealth ratio (second row) with varying Ψ ∈ {0.5,1,2}. The left panels recognize CA and the right are CB.



Finally, we study the sensitivity of robustness preference parameter ψ on the AEW. Two cases are considered. In the first case, if we stick to the baseline EIS $\phi = 0.5$ which is smaller than one, which indicates a retiree's preference for more stable consumption pattern over time, then the AEW is increasing with the retiree's robustness preference. In another case where $\phi = 1.5$, which is greater than one, indicating that a retiree has higher tolerance to future consumption fluctuations, then the relationship between ψ and AEW is reversed. In other words, the aversion of future consumption fluctuations caused by mortality model misspecification can be translated into a fear of longevity risk. Thereby, a growing concern about mortality model uncertainty will essentially cause the AEW to increase. It is also worth mentioning that an AEW[4] as high as 190% as appeared in Figure 5 by no means present flaws in our model or calculation, and it is caused by the fact that LCM is based on a utility maximization framework. It is known that utility function is non-linear and concave, so an immoderate amount of wealth is needed for compensating the utility reduction due to the absence of annuity within the CB market. Indeed, such a magnitude of AEW is consistent with numerous results reported in the literature (see e.g., Brown, 2001; Milevsky and Huang, 2018). Based on the pattern observed in Figure 5, we argue that mortality model uncertainty is a potential contributor to the enduring puzzle of low annuity demand. Namely, if the uncertainty surrounding a point estimate of the mortality curve is overlooked by the retiree, then the value of annuity may be understated when $\phi < 1$, which is a realistic choice for the EIS parameter. Our study acknowledges that retirees should not place a full conviction on a specific mortality assumption. Otherwise, the longevity risk inherent in retirement planning will be underestimated. Educating investors to recognize the uncertainty around future mortality pathways may be one of the possible ways to resolve the issue of low annuity demand in the present retirement market. Moreover, we note that the AEW of the female retiree is always lower than that of the male retiree. The reason is that the female retiree has a lower mortality rate than the male, so the corresponding annuity price is higher, lowering the utility gained.

Figure 5 THE AEW'S WHEN THE EIS Φ = 0.5 (LEFT PANEL) AND Φ = 1.5 (RIGHT PANEL).



[4] Recall that AEW quantifies the extra amount of initial wealth needed in order to compensate for the lack of annuity in the CB market. For example, a retiree in the CB market would need to have an initial wealth of $190 in order to achieve the same level of utility as a retiree in the CA market who only has $100 initial wealth. As long as AEW is greater than 1, that implies annuitization can generate a utility increment. However, note that in our study, we do not try to address the question about whether 190% AEW is good enough. Instead, our goal is to investigate if the presence of mortality model uncertainty makes the baseline AEW smaller or larger. Since we assumed that annuity prices are actuarially fair, the AEW is mainly driven by the risk aversion and the ambiguity aversion of the retiree.

Section 3: A Recap of Yaari's Lifecycle Model

In a nutshell, Yarri (1965) studied the optimal annuity payout/consumption pattern for a utility maximizing retiree facing stochastic lifetime. The annuitization argument derived in the original paper of Yaari (1965) is rather subtle, thus before putting our paper into perspective, this section provides a coarse review of it. Consider a rational retiree aged y at time 0, and non-negative random variable (RV) τ_y denotes the retiree's remaining lifespan. Let λ_{y+t} be the subjective force of mortality of the retiree at time $t \geq 0$, the corresponding survival probability can be computed via

$$ {}_t p_y := \mathbb{P}(\tau_y > t) = exp\left(-\int_0^t \lambda_{y+s}\, ds\right). \tag{2} $$

Suppose that the retiree does not have any bequest motive and is neither willing nor able to invest in the stock market. As a matter of choice, the retiree can either invest the retirement savings in a risk-free bond or an annuity, and then fully consume the payments generated from the holdings. In a complete annuity (CA) market which refers to the availability of a complete set of annuities at actuarially fair prices, the rational retiree will convert all the retirement savings into an annuity. An initial retirement wealth of $x_0 > 0$ can support the annuity payout function $c_A: \Re_+ \rightarrow \Re_+$, such that

$$ x_0 = \mathbb{E}[\int_0^{\tau_y} e^{-rs}\, c_A(s) ds] = \int_0^{\infty} e^{-rs}\ {}_s p_y c_A(s) ds = \int_0^{\infty} e^{-\int_0^s (r+\lambda_{y+u}) du}\, c_A(s) ds, \tag{3} $$

where $r > 0$ denotes the instantaneous risk-free interest rate. The time t actuarial present value of the future annuity payments can be evaluated as

$$ X_A(t) = \mathbb{E}_t[\int_t^{\tau_y} e^{-r(s-t)}\, c_A(s) ds] = \int_t^{\infty} e^{-\int_t^s (r+\lambda_{y+u}) du}\, c_A(s) ds, \tag{4} $$

which satisfies the following differential equation:

$$ dX_A(t) = ((r + \lambda_{y+t}) X_A(t) - c_A(t)) dt, \quad X_A(0) = x_0. \tag{5} $$

Without other means of living, the individual's wealth trajectory is exactly $\{X_A(t)\}_{t\geq 0}$. Here and thereafter, the subscript ``$A$'' attached with the payout function and wealth process is used to emphasize the CA market assumption. (Similarly, we should use subscript ``B'' to spell out the complete bond market condition which will be introduced in a moment.)

For $\gamma > 0$ and $c > 0$, let $u(c) = c^{1-\gamma}/(1-\gamma)$ denote the CRRA utility of consumption[5] (when $\gamma = 1$, the utility function can be understood as $u(c) = log\ c$). The rational retiree will choose an annuity payout for which $c(\cdot)$ maximizes the discounted lifetime utility over consumption:

$$ \mathbb{E}[\int_0^{\tau_y} e^{-\rho s}\, u(c(s)) ds] = \int_0^{\infty} e^{-\int_0^s (\rho+\lambda_{y+u}) du} \times \frac{c(s)^{1-\gamma}}{1-\gamma} ds, \tag{6} $$

where $\rho > 0$ is the subjective discount rate that may or may not be equal to the risk-free interest rate. The optimal annuity payout function, or equivalently the optimal consumption path, is solved to be

[5] The LCM considered in Yaari (1965) used a more general additive utility form, but the choice of CRRA utility in our paper simplifies the present.

$$c_A^*(t) = x_0 \times \frac{[exp(-t(\rho - r))]^{1/\gamma}}{\int_0^\infty {}_t p_y [exp(-\rho s)\, exp(-rs)^{\gamma-1}]^{1/\gamma} ds}. \tag{7}$$

In analogy to the CA market, it is the complete bond (CB) market wherein pure discount bonds are available for any maturities, but annuities are absent. Thus, the retiree has to rely on bonds as the only means of investment. In this case, we denote the bond payout function by $c_B: \mathfrak{R}_+ \to \mathfrak{R}_+$ which satisfies the following budget constraint:

$$x_0 = \int_0^\infty e^{-rs}\, c_B(s) ds. \tag{8}$$

The evolution of the corresponding wealth trajectory is given by

$$dX_B(t) = (rX_B(t) - c_B(t))dt, \quad X_B(0) = x_0.$$

The rational retiree will use the same objective function (5) to derive the optimal retirement consumption path which can be computed via

$$c_B^*(t) = x_0 \times \frac{[{}_t p_y\, exp(-t(\rho - r))]^{1/\gamma}}{\int_0^\infty [{}_t p_y\, exp(-\rho s)\, exp(-rs)^{\gamma-1}]^{1/\gamma} ds}. \tag{9}$$

When $\gamma > 1$, then $c_A^*(t)/X_A^*(t) \geq c_B^*(t)/X_B^*(t)$ for all $t > 0$, where X_A^* and X_B^* denote the wealth processes associated with the optimal consumption rules c_A^* and c_B^*, respectively. The inequality implies that by annuitization, the optimal consumption rate out of the present wealth is higher at all times when the retiree is alive, which leads to the conclusion that the rational retiree should convert all the retirement savings into an annuity upon retirement.

Moreover, to quantify the amount of individual welfare gained by annuitization, the annuity equivalent wealth (AEW) can be used. It is defined through

$$V_A^*(x_0) = V_B^*(\text{AEW}),$$

where V_A^* and V_B^* denote the discounted lifetime consumption utility functions under the CA and CB markets with optimal consumption rules (6) and (8), respectively. The AEW indicates the amount of extra initial wealth needed to compensate the absence of annuity in the CB market. In Yaari's LCM, the AEW is given by

$$\text{AEW} = x_0 \times \left[\frac{\int_0^\infty [{}_t p_y\, exp(-\rho s)\, exp(-rs)^{\gamma-1}]^{1/\gamma} ds}{\int_0^\infty {}_t p_y [exp(-\rho s)\, exp(-rs)^{\gamma-1}]^{1/\gamma} ds}\right]^{\frac{\gamma}{\gamma-1}},$$

which is greater than or equal to x_0 for all $\gamma > 0$, meaning that the retiree would need a larger amount of retirement wealth in the CB market in order to achieve the same level of utility as in the CA market.

Section 4: Formulation of the LCM with Mortality Model Uncertainty

As mentioned in the introduction, the main goal of our paper is to advance Yaari's LCM along two directions, namely mortality model uncertainty and recursive utility. We start off by introducing a continuous-time version of the Epstein-Zin recursive utility (Duffie and Epstein, 1992; Garleanu and Panageas, 2015) to model the retiree's preferences. For any $t > 0$, define the actuarial subjective discount factor $\delta_t = \rho + \lambda_{y+t}$, and let $V(t)$ be the discounted future utility at time t. The recursive utility is defined as

$$V(t) = \mathbb{E}_t[\int_t^{\infty} f\,(c(s), \delta_s, V(s))ds], \tag{10}$$

where $c(\cdot)$ denotes the consumption rate, $\mathbb{E}_t[\cdot]$ denotes the conditional expectation given the filtration $\mathcal{F}_t$, and

$$f(c,\delta,v) = \frac{(1-\gamma)v}{1-1/\phi}\left[\left(\frac{c}{((1-\gamma)v)^{\frac{1}{1-\gamma}}}\right)^{1-1/\phi} - \delta\right] \qquad for\ c,\delta,v > 0 \tag{11}$$

is known as the normalized aggregator of consumption and utility. In the formula above, $\gamma > 0$ is the relative risk aversion coefficient which measures the retiree's aversion of consumption fluctuations due to the random states of the future, and $\phi > 0$ is the EIS coefficient which measures the aversion of consumption fluctuations over time in a deterministic world. Hence, the merit for adopting the recursive utility over the CRRA utility is spelled out in that the retiree's preferences over the timing of the resolution of uncertainty is disentangled from risk aversion, so that we can study them separately.

It is noteworthy that the recursive utility specified in (9) actually includes the CRRA utility as a special case, which is summarized in the following remark.

Remark 1. If $\phi = 1/\gamma$, then the aggregator defined in (10) becomes

$$f(c,\delta,v) = \frac{c^{1-\gamma}}{1-\gamma} - \delta v,$$

so the recursive discounted utility (9) reduces

$$V(t) = \mathbb{E}_t[\int_t^{\infty} e^{-\int_t^s \delta_u du} \times \frac{c(s)^{1-\gamma}}{1-\gamma} ds], \tag{12}$$

which coincides with the additive discounted utility (5) as in Yaari's LCM. However, $\phi = 1/\gamma$ may not hold true in general and thus the recursive utility is not necessarily additive.

Next we turn to specify a set of plausible probability measures in order to account for the uncertainty around the reference mortality model. Following the mortality ambiguity approach proposed in Shen and Su (2019), for any $t > 0$, we introduce an $\mathcal{F}_t$-predictable process $\theta(t) > 0$ to be chosen endogenously by the retiree for adjusting the reference subjective mortality curve. Consider an equivalent probability measure $\mathbb{Q}$ which is defined via the Radon–Nikodym's derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = exp\{\int_0^{t\wedge\tau_y}[\,log(\,\theta(s)) - \theta(s) + 1]\lambda_{y+s}ds + \int_0^t log(\,\theta(s))\;dZ(s)\,\} \tag{13}$$

where $Z(s) := \mathbf{1}_{\{\tau_y \le s\}} - \int_0^s \mathbf{1}_{\{\tau_y > u\}}\,\lambda_{y+u}du$ is a martingale associated with the single jump process $\mathbf{1}_{\{\tau_y \le s\}}$. By Girsanov's Theorem, from $\mathbb{P}$ to $\mathbb{Q}$, the subjective force of mortality is perturbed from λ_{y+t} to $\lambda^{\mathbb{Q}}_{y+t} = \theta(t)\lambda_{y+t}$, $t > 0$. Hence, we refer to $\mathbb{Q}$ as the perturbed measure and $\theta(\cdot)$ as the mortality perturbation function. The corresponding survival probability under the perturbed measure $\mathbb{Q}$ can be computed similarly to Equation (1), but with the force of mortality therein replaced by $\theta(t)\lambda_{y+t}$. That is,

$$ {}_t p_y^{\mathbb{Q}} := \mathbb{Q}[\tau_y > t] = exp\left(-\int_0^t \theta(s)\lambda_{y+s} ds\right). \tag{14}$$

The perturbation function $\theta(t)$ manipulates the discrepancy between the alternative model and the reference model at each time $t > 0$. To quantify the overall discrepancy, the relative entropy is a commonly used statistic to measure distance, suitable for robust control problems. In the context of this current paper, the relative entropy between the perturbed mortality model and the best-estimated reference mortality model can be computed via

$$\mathcal{D}(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_t^{\mathbb{Q}}[log(\frac{d\mathbb{Q}}{d\mathbb{P}})] = \mathbb{E}^{\mathbb{Q}}[\int_0^{t\wedge\tau_y}[log(\theta(s)) - \theta(s) + 1]\lambda_{y+s} ds + \int_0^t log(\theta(s))\, dZ(s)]$$

$$= \mathbb{E}^{\mathbb{Q}}[\int_0^{t\wedge\tau_y}[\theta(s)\, log(\theta(s)) - \theta(s) + 1]\lambda_{y+s} ds + \int_0^t log(\theta(s))\, dZ^{\mathbb{Q}}(s)]$$

$$= \mathbb{E}^{\mathbb{Q}}[\int_0^{t\wedge\tau_y}[\theta(s)\, log(\theta(s)) - \theta(s) + 1]\lambda_{y+s} ds].$$

For notational convenience, letting

$$g(\theta) := \theta\, log\, \theta - \theta + 1, \quad \theta > 0,$$

then we can write

$$\mathcal{D}(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}[\int_0^{t\wedge\tau_y} g(\theta(s))\lambda_{y+s} ds]. \tag{15}$$

It is straightforward to check that $g(1) = 0$ and $g'(\theta) = log\, \theta$, so if $\theta(t) \equiv 1$, then the perturbed mortality model coincides with the reference mortality model, and so the relative entropy $\mathcal{D} = 0$. For θ_1 and θ_2 both smaller than or both greater than one, if $|\theta_1(t) - 1| > |\theta_2(t) - 1|$ for all $t > 0$, then the perturbed mortality curve associated with θ_1 is farther away from the reference curve than the one associated with θ_2, so the corresponding entropy satisfies $\mathcal{D}_1 > \mathcal{D}_2$.

The discussion above concerns how to construct the set of alternative mortality models to be considered by the retiree. Another important question is how large the set of the alternative mortality models should be. The answer to this question is not unique, but a widely accepted principle is that alternative models should not be statistically too far away from the reference model which has been best estimated, and those alternative models that are hard-to-be-distinguished from the reference model should be considered more seriously. The penalty approach employed in the literature, such as Maenhout (2004, 2006) and Shen and Su (2019), reflects the aforementioned principle. Formally, given the consumption function $c(\cdot)$ and perturbation function $\theta(\cdot)$, we specify a penalty term incurred to the retiree's discounted utility such that

$$J(c, \theta; t, x) = \mathbb{E}_{x,t}^{\mathbb{Q}}[\int_t^{\infty} f(c(s), \delta_s, V(s, X(s)))ds] + \frac{1}{\psi} \times \Gamma(t, \theta), \quad t > 0,\ X(t) = x, \tag{16}$$

where $\psi > 0$ is the robustness preference parameter reflecting the extent of retiree's concern about the uncertainty surrounding the reference mortality model, and

$$\Gamma(t, \theta) = \mathbb{E}^{\mathbb{Q}}[\int_0^{\infty}(1 - \gamma)\, V(s, X(s))g(\theta(s))\lambda_{y+s} ds]$$

can be viewed as a scaled counterpart of the relative entropy given in (14). The scaling factor $(1 - \gamma)V(s, X(s))$ is included in the penalty term mainly for an analytic tractability reason. With a fixed ψ, since a perturbed model that is far away from the reference model will cause a large penalty to the utility, such a perturbed model is less likely to be accepted by the retiree. Alternatively, the robustness preference parameter ψ controls the set of the alternative mortality models that the retiree is willing to consider. Namely, as ψ increases, the penalty term becomes smaller even for those alternative models that are significantly different from the reference model, so the set of alternative mortality models acceptable for the retiree expands. In other words, a more robust retiree having higher ψ will put less faith on the reference model and effectively consider more different alternative mortality models that possess larger relative entropies. When $\psi = 0$, any deviation from the reference model will lead to an infinitely large penalty incurred to the utility function. Hence in this case, the retiree has no ambiguity aversion and fully trusts the reference mortality model, corresponding to Yaari's LCM under the recursive utility but without mortality model uncertainty.

Given the mortality perturbed measure and recursive utility, the retiree's robust decision-making problem can be formulated in terms of the following two contemporaneous courses of action. In one course of action, within the pool of plausible mortality models, the retiree seeks to identify the worst-case mortality perturbation function $\theta^*(\cdot)$ that is most unfavorable to the retiree's consumption utility. In another course of action, the retiree selects the optimal consumption policies to maximize the recursive utility under the worst-case mortality scenario. Collectively, the value function associated with the retirement problem of interest reads as

$$V(t,x) = \max_{c\in \mathrm{C}} \min_{\theta\in \mathrm{T}} J(c,\theta;t,x), \qquad t > 0 \ and \ x > 0 \tag{17}$$

where the objective function $J(\cdot)$ is defined as per (15), $\mathfrak{C}$ and $\mathfrak{T}$ are the admissible spaces for consumption strategies and perturbation functions, to be specified in Definition 1 below.

Definition 1. A consumption strategy $c(t)$ is said to be admissible if

- $c(t) \geq 0$, for all $t > 0$;
- $\int_0^\infty c\,(s)ds < \infty$;
- the wealth process $X(t)$ associated with $c(t)$ stays positive over the entire planning horizon.

The space of all admissible consumption strategies is denoted by $\mathfrak{C}$.

Moreover, a distortion process $\theta(t)$ is said to be admissible if

- $\theta(t) > 0$, for all $t > 0$;
- $\mathbb{Q}$ is a well-defined probability measure equivalent to $\mathbb{P}$.

The space of all admissible distortion processes is denoted by $\mathfrak{T}$.

Inspired by the original study of LCM in Yaari (1965), we consider both the CA and CB market conditions, under which the retirement problem (16) satisfies the budget constraints specified in Equations (2) and (7), respectively. Capitalizing on the extended Yaari's LCM laid down in this current section, we have a keen interest in examining the following questions which are of great importance in modern retirement research:

Q1. If mortality model uncertainty is concerned, what will be the worst-case mortality scenario for the retiree? (See Points 2 and 3 in Section 2.2 for a concise answer, and Section 5.1 for detailed discussions.)

Q2. Under mortality model uncertainty, what will be the optimal consumption policies? (See Point 4 in Section 2.2 for a concise answer, and Section 5.1 for detailed discussions.)

Q3. What are the implications of mortality model uncertainty on the AEW and the annuity puzzle? (See Point 6 in Section 2.2 for a concise answer, and Section 5.2 for detailed discussions.)

Section 5: Main Results

Table 1 summarizes the cases that we aim to investigate and compare in this section. Among the four cases, Cases A and B have been well studied in Yaari (1965) under the additive utility function, but we extend the study to the more general recursive utility. The first part of this section studies the optimal consumption policies under Cases C and D, thus answering questions Q1 and Q2. Comparisons between Case A and Case C, and between Case B and Case D, are considered in the second part of this section, which answers question Q3.

Table 1

SUMMARY OF THE FOUR DIFFERENT RETIREMENT CASES CONSIDERED IN THIS CURRENT PAPER

	Ambiguity neutral	Ambiguity averse
Complete annuity market	Case A	Case C
Complete bond market	Case B	Case D

5.1 OPTIMAL STRATEGIES WITH MORTALITY MODEL UNCERTAINTY

In this section, we apply the dynamic programming principle to solve the max-mix problem (16), which yields the robust optimal consumption strategies for the rational retiree described in Section 4. The succeeding theorem summarizes the main mathematical results. Recall that quantities related to the CA and CB markets are distinguished by the subscripts ``$A$'' and ``B'', respectively. The robustness preference parameter ψ is specified in the optimal decision rules in order to highlight the retiree's aversion against mortality model uncertainty which constitutes a major object of interest in our paper.

Theorem 1. Suppose the retirement environment as per the description in Section 4, the worst-case perturbation function associated with the optimization problem (16), can be computed via

$$\theta_A^*(t;\psi) = \theta_B^*(t;\psi) \equiv \theta^*(\psi) = exp(\tfrac{\psi}{1-1/\phi}), \qquad t > 0,$$

where $\psi > 0$ denotes the robust preference parameter and $\phi > 0$ denotes the EIS coefficient. Moreover, the optimal robust consumption strategies are given by

$$c_\square^*(t;\psi) = c_\square^*(0;\psi) \times exp\{\int_0^t [(1 - G_\square(\psi))\lambda_{y+u} - \phi(\rho - r)]du\}, \tag{18}$$

where "$\square$" can be "A" or "B", and

$$G_A(\psi) := (1-\phi) + G_B(\psi), \quad G_B(\psi) := \phi\theta^* + \frac{1-\phi}{\psi} g(\theta^*).$$

Moreover, the optimal initial consumption rate can be evaluated via

$$c_\square^*(0;\psi) = \frac{x_0}{K_\square(\psi)}, \qquad with \;\; K_\square(\psi) = \int_0^\infty exp\{-\int_0^s (\beta + G_\square(\psi)\lambda_{y+u})\,du\}ds, \tag{19}$$

in which $\beta = (1-\phi)r + \phi\rho$ is the weighted average between the risk-free interest rate and subjective discount rate.

Given the optimal robust strategies $c_\square^*$ and θ^*, the value function (16) at the present time can be computed via

$$V_{\square}^{*}(0,x_0;\psi)=[K_{\square}(\psi)]^{-\frac{1-\gamma}{1-\phi}}\frac{x_0^{1-\gamma}}{1-\gamma},$$

where "□" is either "A" or "B".

Proof. See Appendix A. □

A careful inspection of the optimal strategies outlined in Theorem 1 reveals the following findings. First, the worst-case perturbation functions are identical under the CA market and CB market conditions. This is because the only difference between the optimization problems under the two market conditions is the associated budget constraints which only depend on the risk-free interest rate parameter and the reference mortality curve, while the worst-case perturbation functions $\theta_A^*(\cdot)$ and $\theta_B^*(\cdot)$ only depend on the EIS coefficient ϕ and robustness preference parameter ψ. Interestingly, the worst-case perturbation functions are constant over time with the optimal perturbed mortality model given by

$$\lambda_{y+t}^{*}=\theta^{*}\times\lambda_{y+t},\quad t>0, \tag{20}$$

which is a parallel shift of the best-estimated reference mortality curve. In the language of actuarial mathematics, the form of modification in (19) is also known as the proportional hazard distortion (Wang, 1996). The proportional transform (a.k.a., parallel shock) is also commonly used in retirement research to test the sensitivity of mortality assumptions (see, e.g., Shen and Su, 2019). The proportional transform is applied in earlier studies mainly because of its inherent simplicity. Our finding herein provides a theoretical justification for the choice of parallel shock in sensitivity tests, which is perhaps sufficiently conservative for covering the worst-case mortality scenario, so far at least as Yaari's LCM is concerned.

Second, the selection of the worst-case mortality shock θ^* depends on the interplay between the EIS coefficient ϕ and robustness preference ψ. Recall that ψ^{-1} conveys the amount of faith that the retiree puts on the reference model. The larger the value of ψ is, the larger the value of $|\theta^*-1|$ becomes, which implies that a retiree with a stronger concern about model uncertainty will rationally set the worst-case perturbed model to be farther away from the reference model. Meanwhile, if $(1-1/\phi)<0$, or equivalently $\phi<1$, then $\theta^*<1$, which corresponds to an improved mortality scenario, and vice versa. Empirical studies have already suggested that the EIS coefficient ϕ for investors is typically less than 1 (e.g., Yogo, 2014). In this sense, the longevity risk is more relevant to the context of this current paper.

What is more, as mentioned in Remark 1, if $1/\phi=\gamma$, then the recursive utility reduces to the power utility. In this case, the optimal perturbation function becomes

$$\theta^{*}(t)=e^{\frac{\psi}{1-\gamma}}. \tag{21}$$

Based on a more complicated lifecycle model, Shen and Su (2019) adopted the same penalty approach to obtain the optimal robust consumption-investment-insurance strategy when there are uncertainties around both the economics and mortality models, and the investor's preference is depicted by a power utility. Though analytical expression is not available for $\theta^*(t)$ due to the mathematical complexity involved in Shen and Su (2019), it was shown that $\theta^*(t)<1$ if the risk aversion parameter of the power utility satisfies $\gamma>1$, and vice versa. In this regard, (20) is consistent with the finding in Shen and Su (2019). The use of recursive utility framework in this current paper allows us to distinguish the EIS coefficient from the risk aversion parameter. As a result, we further clarify the assertion in Shen and Su (2019) by theorizing that whether the worst-case perturbed mortality curve corresponds to a mortality risk scenario (i.e., $\theta^*>1$) or a longevity risk scenario (i.e., $\theta^*<1$) depends solely on the EIS parameter, but not the risk aversion parameter. This is one of the major economic implications of our paper. Our discussion thus far in this current section answers question Q1 posted at the end of Section 4.

Next, let us focus on the optimal consumption strategies. Curiously, the optimal consumption function (17) is independent of the risk aversion parameter γ of the recursive utility. To see the reason, recall that γ captures the aversion against consumption fluctuation due to the uncertain state in the future. As the retiree has converted the retirement savings into either annuity or bond investment at the beginning of the planning horizon[6], there is no uncertainty involved in the future cash flows. Consequently, the optimal consumption strategies $c_A^*(\cdot)$ and $c_B^*(\cdot)$ depend only on the EIS coefficient, together with other parameters including mortality and discount rates, but not the risk aversion parameter γ.

It is also interesting to study the patterns of the optimal consumption pathways over time. The succeeding corollary summarizes the increasing and decreasing properties for the optimal consumption function (17).

Corollary 2. For $\square \in \{A, B\}$ and any $t \geq 0$, if $(1 - G_\square(\psi))\lambda_{y+t} \geq \phi(\rho - r)$, then the optimal consumption $c^*(t;\psi)$ is increasing in t, and vice versa.

Proof. The proof follows immediately from the expression of optimal consumption function (17). $\square$

The following lemma is of auxiliary importance in our latter discussion.

Lemma 3. For all $\psi > 0$, the functions $G_A(\psi)$ and $G_B(\psi)$ are decreasing in ψ if the EIS $\phi < 1$, and increasing in ψ otherwise. Further, if $\phi < 1$, then

$$1 - \phi \leq G_A(\psi) \leq 1 \quad \text{and} \quad 0 \leq G_B(\psi) \leq \phi.$$

Otherwise,

$$1 \leq G_A(\psi) \leq \infty \quad \text{and} \quad \phi \leq G_B(\psi) \leq \infty.$$

Proof. See Appendix A. $\square$

Corollary 2 and Lemma 3 together imply that the optimal consumption pathways may present an asymmetric U-shaped pattern over time (Figure 4). For instance, consider a realistic situation in which EIS $\phi < 1$ and $\rho > r$, then the optimal consumption function may decrease over time at the beginning of the retirement phase when the force of mortality is still relatively low and so $(1 - G_\square(\psi))\lambda_{y+t} \leq \phi(\rho - r)$, but it will essentially become increasing in time as mortality rate grows during the later stage of retirement.

The next assertion compares the optimal consumption rules between the CA market and the CB market (i.e., Case C versus Case D in Table 1). Let $\pi(t) = c(t)/X(t)$ be the consumption-to-wealth ratio at time $t > 0$, which indicates the retiree's propensity to consumption out of the present wealth. The consumption-to-wealth ratio associated with the optimal robust consumption strategy reported in Theorem 1 is denoted by $\pi_\square^*(t;\psi) = c_\square^*(t;\psi)/X_\square^*(t;\psi)$, where $X_\square^*(\cdot\,;\psi)$ is the corresponding wealth process, $\square \in \{A, B\}$.

Proposition 4. The following relationships hold for the optimal strategies identified in Theorem 1:

1. If the EIS $\phi \begin{Bmatrix} < \\ > \end{Bmatrix} 1$, then $\pi_A^*(t;\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} \pi_B^*(t;\psi)$ for all $t > 0$ and $\psi > 0$.
2. If the EIS $\phi < 1$ (resp. $\phi > 1$), then there exists a time epoch $t^* > 0$ such that $c_A^*(t;\psi)$ is greater (resp. smaller) than or equal to $c_B^*(t;\psi)$ for $t \leq t^*$, but the inequality is reversed for $t > t^*$.

[6] Compare between the CA and CB markets, it is rational to purchase an annuity rather than a bond investment regardless of the risk aversion level of the retiree. Thereby, when making the investment decision, the rationality of the retiree comes into play rather than the risk aversion level.

3. For any EIS $\phi > 0$ and robustness preference $\psi > 0$, $V_A^*(0, x_0; \psi) > V_B^*(0, x_0; \psi)$.

Proof. See Appendix A. □

Proposition 4 answers question Q2 posted in Section 4. Specifically, it shows that even with mortality model uncertainty, annuitization may increase the optimal consumption rate at all times if the EIS parameter $\phi < 1$which is a realistic case. Although there is a twisted pattern in the comparison in terms of the absolute consumption amount, the discounted lifetime utility of consumption is always higher by purchasing an annuity.

In concluding this subsection, we report another important component in the study of Yaari's LCM, namely the AEW, but in the state of mortality model uncertainty and recursive utility.

Theorem 5. Suppose the retirement environment as per the description in Section 4, and the optimal decision rules can be computed via Theorem 1. Given the initial wealth $x_0 > 0$, the associated annuity equivalent wealth is

$$\mathrm{AEW}(\psi) = x_0 [\frac{K_B(\psi)}{K_A(\psi)}]^{1/(1-\phi)}, \quad \psi > 0.$$

Proof. See Appendix A. □

As mentioned earlier, because the constraint of full annuitization in the CA market or full bond investment in the CB market removes the uncertainty in the future cash flows, the AEW does not depend on the risk aversion parameter γ which captures the consumption fluctuation due to future uncertainty.

Remark 2. Note that the AEW reported in Theorem 5 is always greater than x_0. To see why, recall that $G_A(\cdot) = (1 - \phi) + G_B(\cdot)$, which implies $[G_A(\cdot)]^{1/(1-\phi)} \geq [G_B(\cdot)]^{1/(1-\phi)}$, and so $[K_A(\cdot)]^{1/(1-\phi)} \geq [K_B(\cdot)]^{1/(1-\phi)}$ according to Equation (18). This inequality confirms again that annuitization will induce extra utility to the retiree even when there is a presence of mortality ambiguity aversion.

5.2 IMPLICATIONS OF MORTALITY MODEL UNCERTAINTY

The prior subsection is devoted to the study of the optimal robust consumption strategies for Yaari's LCM equipped with the recursive utility. In this section, we are going to focus on the implications of mortality model uncertainty on the optimal decision rules as well as the AEW.

Proposition 6. For any fixed $t > 0$ and $\square \in \{A, B\}$, the optimal consumption-to-wealth ratio $\pi_\square^*(t; \psi)$ is decreasing in the robustness preference parameter ψ if the EIS coefficient $\phi < 1$, or increasing otherwise.

Proof. See Appendix A. □

The above result shows that when EIS $\phi < 1$, the mortality ambiguity aversion will lower the percentage of consumption out of the present wealth at every instant, no matter whether or not an annuity is purchased. This is because if $\phi < 1$, then the worst-case perturbation function satisfies $\theta^* < 1$, which corresponds to a longevity risk scenario. Consequently, the retiree reduces the consumption rate in order to lower the risk of outliving retirement savings. Alternatively, if EIS $\phi > 1$, then the worst-case perturbed mortality curve corresponds to a mortality risk scenario. So in order to maximize the lifespan discounted utility, the rational retiree will choose to increase the consumption ratio.

In addition to the relative consumption ratio, we are also interested in studying the impact of mortality model uncertainty on the absolute consumption amount. Intuitively, if the robustness preference parameter ψ changes, the optimal robust consumption rules should decrease for some t and increase for the others, so that the budget constraint (2) or (7) can be satisfied. The next assertion shows that in the case of $\phi < 1$ where the longevity risk is more concerned, the retiree will rationally reduce the consumption amount at the beginning of the retirement phase so as to keep more savings for the future. In another case with EIS $\phi > 1$, and thus the mortality risk is more concerned, the rational retiree will choose to increase the consumption amount at the beginning of the retirement phase so as to make sure that a desirable amount of consumption utility can be gained before death.

Proposition 7. For $\square \in \{A, B\}$, there exists a time epoch $t^* > 0$ such that the optimal consumption functions $c_\square^*(t;\psi)$ reported in Theorem 1 are decreasing in ψ for all $t \leq t^*$ and become increasing in ψ for $t > t^*$, if the EIS $\phi < 1$. Otherwise, the behaviors of the optimal consumption functions are reversed.

Proof. See Appendix A. $\square$

Remark 3. Consider the case where $\psi \to 0$ meaning that the retiree has no ambiguity aversion against the mortality model, then

$$\lim_{\psi \to 0} G_A(\psi) = 1 \quad \text{and} \quad \lim_{\psi \to 0} G_B(\psi) = \phi.$$

The optimal consumption functions become

$$\lim_{\psi \to 0} c_A^*(t;\psi) = x_0 \times \frac{exp(-\int_0^t \phi\,(\rho - r)du)}{\int_0^\infty exp\{-\int_0^s (\beta + \lambda_{y+u})\,du\}ds} \tag{22}$$

and

$$\lim_{\psi \to 0} c_B^*(t;\psi) = x_0 \times \frac{exp\{-\int_0^t\ [\phi(\rho - r + \lambda_{y+u}) - \lambda_{y+u}]du\}}{\int_0^\infty exp\{-\int_0^s \left(\beta + \phi\lambda_{y+u}\right)du\}ds}. \tag{23}$$

Further, suppose that the EIS and risk aversion parameters satisfy $\phi = 1/\gamma$ so the recursive utility reduces to the power utility, then (21) and (22) collapse respectively to the optimal consumption functions (6) and (8) under the classic Yaari's LCM.

Finally, the impact of mortality model uncertainty on the AEW is considered. Recall again that if EIS $\phi < 1$, then the associated worst-case perturbed mortality curve corresponds to a longevity risk scenario, and as ψ increases, the concern about longevity risk becomes stronger, so an annuity should become more valuable. In other words, AEW increases with the robustness preference parameter ψ when the EIS $\phi < 1$, and vice versa. The succeeding assertion confirms our conjecture.

Proposition 8. If $\phi < 1$ (resp. $\phi > 1$), then $\mathrm{AEW}(\psi)$ is increasing (resp. decreasing) in $\psi > 0$.

Proof. See Appendix A. $\square$

The study in this current subsection answers question Q3 posted in Section 4.

Section 6: Conclusions

In this paper, we proposed and studied a revamped LCM in which there is an incorporation of mortality model uncertainty. We derived the optimal robust consumption decision as well as the associated AEW in explicit forms. Our major economic findings include the following. First, we found that for a typical retiree having EIS smaller than one, i.e., a more risk averse retiree, the worst-case perturbed mortality curve corresponds to an improved mortality scenario, meaning that the longevity risk is more of a concern than mortality risk in retirement planning. Second, under mortality model uncertainty, an annuity should still be attractive to retirees in the sense that by annuitization, the optimal consumption rate becomes higher. However, mortality ambiguity aversion will lower the optimal consumption rate. Third, if mortality model uncertainty is ignored by the retiree, then the value of annuity will be understated, causing a lower-than-expected annuity demand.

There are several topics for future research. First, as pointed out by one of the project oversight group members, it is likely that a retiree's ambiguity aversion level may be dependent of her/his risk aversion level. It will be very interesting to explore what are the common determinants of the two different types of aversion behaviors. Second, it will be interesting to investigate whether or not retirees actually behave rationally. Third, inflation has been growing up rapidly over the past two years. It will be interesting to incorporate a stochastic inflation rate model into the analysis and study the impacts of inflation jumps on the optimal drawdown strategies. Another promising research question needed to be addressed is how to best educate retirees, who may lack quantitative backgrounds, to understand and acknowledge model uncertainty.

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Appendix A: Proofs

Proof of Theorem 1. We only provide the proof for the CA market. The proof for the CB market is essentially the same, thereby omitted. To simplify our notation, we suppress the subscript A in V and K in the sequel.

Given $X_t = x > 0$, the Hamilton–Jacobi–Bellman (HJB) equation for the optimization problem (16) can be specified as

$$\max_{c\in C}\min_{\theta\in T}\left\{V_t + V_x\left[(r+\lambda)x - c\right] + \frac{(1-\gamma)V}{1-1/\phi}\left[\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-1/\phi} - (\rho+\theta\lambda)\right] + \frac{1-\gamma}{\psi}g(\theta)\lambda V\right\} = 0.$$

By the first order conditions of C and θ, we get

$$-V_x + \frac{(1-\gamma)V}{1-1/\phi}\frac{1-1/\phi}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{-1/\phi} = 0 \tag{24}$$

and

$$-\lambda\frac{(1-\gamma)V}{1-1/\phi} + \frac{1-\gamma}{\psi}g'(\theta)\lambda V = 0. \tag{25}$$

We conjecture the following *ansatz*

$$V(t,x;\psi) = [K(t;\psi)]^{-\frac{1-\gamma}{1-\phi}}\frac{x^{1-\gamma}}{1-\gamma}$$

with the terminal value of K is $K(\infty;\psi) = 0$, is a solution to the HJB equation above. Then from Equations (24) and (25), we obtain

$$c^*(t;\psi) = \frac{x}{K(t;\psi)} \quad \text{and} \quad \theta^*(t;\psi) \equiv \theta^*(\psi) = e^{\frac{\psi}{1-1/\phi}}. \tag{26}$$

To solve K, we substitute c^* and θ^* back to the HJB equation and get

$$-\frac{1}{1-\phi}K_t(t;\psi)\,[K(t;\psi)]^{-\frac{1-\gamma}{1-\phi}-1}x^{1-\gamma} + [K(t;\psi)]^{-\frac{1-\gamma}{1-\phi}}x^{1-\gamma}[(r+\lambda) - \frac{1}{K(t;\psi)}] + \frac{[K(t;\psi)]^{-\frac{1-\gamma}{1-\phi}}x^{1-\gamma}}{1-1/\phi}[\frac{1}{K(t;\psi)} - (\rho+\theta^*(\psi)\lambda)] + \frac{\lambda}{\psi}g(\theta^*(\psi))[K(t;\psi)]^{-\frac{1-\gamma}{1-\phi}}x^{1-\gamma} = 0.$$

Standard algebraic manipulation yields

$$K_t(t;\psi) - (1-\phi)[(r+\lambda) - \frac{\rho+\theta^*(\psi)\lambda}{1-1/\phi} + \frac{\lambda}{\psi}g(\theta^*(\psi))]K(t;\psi) + 1 = 0,$$

whose solution is obviously given by

$$K(t;\psi) = \int_t^\infty exp\{-\int_t^s(\beta + G(\psi)\lambda_{y+u})\,du\}ds.$$

Denote by $X^*(\cdot)$ the wealth trajectory associated with the optimal consumption derived above, then the dynamic of the optimal consumption c^* evolves as

$$dc^*(t;\psi) = \frac{1}{K(t;\psi)}[-(\beta + G(\psi)\lambda_{y+t}) + (r + \lambda_{y+t})]X^*(t;\psi)dt$$

$$= [(1 - G(\psi))\lambda_{y+t} - \phi(\rho - r)]c^*(t;\psi)dt.$$

Hence,

$$c^*(t;\psi) = c^*(0;\psi)\, exp\{\int_0^t [(1 - G(\psi))\lambda_{y+u} - \phi(\rho - r)]du\}.$$

The proof is now completed. □

Proof of Lemma 3. First, we consider the monotonicity for G_A and G_B. Standard algebraic manipulations yield

$$\frac{\partial}{\partial\psi}G_A(\psi) = \frac{\partial}{\partial\psi}G_B(\psi) = \frac{1-\phi}{\psi^2}[e^{\frac{\psi}{1-1/\phi}}(1 - \frac{\psi}{1-1/\phi}) - 1].$$

Note that $e^a(1 - a) < 1$ for any $a \in (-\infty, \infty)$. We have, for $\square \in \{A, B\}$,

$$\frac{\partial}{\partial\psi}G_\square(\psi) = \begin{cases} > 0, & if\ \phi < 1; \\ < 0, & if\ \phi > 1. \end{cases}$$

Next, let us focus on G_A, and we have

$$\lim_{\psi\to 0} G_A(\psi) = 1 - \phi + \lim_{\psi\to 0}\frac{1-\phi}{\psi}(1 - e^{\frac{\psi}{1-1/\phi}}) = 1$$

and

$$\lim_{\psi\to\infty} G_A(\psi) = 1 - \phi + \lim_{\psi\to\infty}\frac{1-\phi}{\psi}\left(1 - e^{\frac{\psi}{1-1/\phi}}\right) = \begin{cases} 1 - \phi, & if\ \phi > 1; \\ \infty, & if\ \phi < 1. \end{cases}$$

Moreover, note that for $G_B(\psi) = G_A(\psi) - (1 - \phi)$, the desired results are readily obtained. The proof is completed.
□

Proof for Proposition 4. We prove the three relationships one by one. For the first relationship, we know from Equation (26) in the proof of Theorem 1 that $\pi_\square^*(t;\psi) = 1/K_\square(t;\psi)$ for a fixed $t > 0$ and $\square \in \{A, B\}$.

For all $\psi > 0$, we readily obtain

$$\phi \begin{Bmatrix} < \\ > \end{Bmatrix} 1 \;\Rightarrow\; G_A(\psi) \begin{Bmatrix} < \\ > \end{Bmatrix} G_B(\psi) \;\Rightarrow\; K_A(t;\psi) \begin{Bmatrix} < \\ > \end{Bmatrix} K_B(t;\psi) \;\Rightarrow\; \pi_A^*(t;\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} \pi_B^*(t;\psi). \quad (27)$$

Also note that the above inequalities yield

$$\phi \begin{Bmatrix} < \\ > \end{Bmatrix} 1 \quad\Rightarrow\quad c_A^*(0;\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} c_B^*(0;\psi).$$

Meanwhile, we have

$$\frac{c_A^*(t;\psi)}{c_B^*(t;\psi)} = \frac{K_B(\psi)}{K_A(\psi)} \times \exp\left\{\int_0^t [G_B(\psi) - G_A(\psi)]\lambda_{y+u}\, du\right\},$$

which is decreasing in $t > 0$ if $\phi < 1$, or increasing otherwise. This yields the second relationship in the proposition.

Another repeated application of the inequalities in (27) yields the third relationship. This completes the proof. □

Proof of Theorem 5. By definition, the AEW is obtained via solving

$$V_A^*(0, x_0; \psi) = V_B^*(0, AEW; \psi).$$

According to Theorem 1, the AEW solves

$$[K_A(\psi)]^{-\frac{1-\gamma}{1-\phi}} \frac{x_0^{1-\gamma}}{1-\gamma} = [K_B(\psi)]^{-\frac{1-\gamma}{1-\phi}} \frac{\mathrm{AEW}^{1-\gamma}}{1-\gamma}.$$

This yields

$$\mathrm{AEW}(\psi) = x_0 [\frac{K_B(\psi)}{K_A(\psi)}]^{1/(1-\phi)},$$

which completes the proof. □

Proof of Proposition 6. From the proof of Proposition 4, we know that $\pi_\square^*(t;\psi) = 1/K_\square(t;\psi)$ for a fixed $t > 0$ and $\square \in \{A, B\}$. Consider

$$\frac{\partial}{\partial\psi} K_\square(t;\psi) = -\frac{\partial}{\partial\psi} G_\square(\psi) \times \int_t^\infty \exp\Big(-\int_t^s (\beta + G_\square(\psi)\lambda_{y+u})du\Big) \int_t^s \lambda_{y+u} du\, ds = \begin{cases} > 0, & if\ \phi < 1 \\ < 0, & if\ \phi > 1. \end{cases}$$

So $\pi_\square^*(t;\psi)$ is decreasing in ψ if the EIS $\phi < 1$, or increasing otherwise. The proof is completed. □

Proof of Proposition 7. For $t > 0$ and $\square \in \{A, B\}$, write

$$c_\square^*(t;\psi) = \frac{x_0}{K_\square(\psi)} \exp\left\{\int_0^t \left[\left(1 - G_\square(\psi)\right)\lambda_{y+u} - \phi(\rho - r)\right] du\right\},$$

so we have

$$\frac{\partial}{\partial\psi} c_\square^*(t;\psi) = \frac{x_0}{\left[K_\square(\psi)\right]^2} \exp\left\{\int_0^t \left[\left(1 - G_\square(\psi)\right)\lambda_{y+u} - \phi(\rho - r)\right] du\right\} \left(-\frac{\partial}{\partial\psi} G_\square(\psi)\right)$$

$$\times \left\{ K_\square(\psi) \int_0^t \lambda_{y+u}\, du - \int_0^\infty \exp\left(-\int_0^s (\beta + G_\square(\psi)\lambda_{y+u})\, du\right) \int_0^s \lambda_{y+u}\, du ds\right\}.$$

Suppose that $\phi < 1$, from Corollary 2, we know $\partial/\partial\psi G_\square(\psi) < 0$. An inspection of the partial derivative formula above reveals

$$\lim_{t\to 0}\frac{\partial}{\partial\psi}c_{\square}^{*}(0;\psi)<0 \quad \text{as well as} \quad \lim_{t\to\infty}\frac{\partial}{\partial\psi}c_{\square}^{*}(t;\psi)>0,$$

and moreover, $\frac{\partial}{\partial\psi}c_{\square}^{*}(t;\psi)=0$ has a unique root. This establishes the desired results when $\phi<1$. If $\phi>1$, then reversed behaviors of the optimal consumption functions are obtained. The proof is now completed. □

Proof of Proposition 8. Recall that by Theorem 5, the AEW can be computed via

$$\mathrm{AEW}(\psi)=x_0[\frac{K_B(\psi)}{K_A(\psi)}]^{1/(1-\phi)}, \quad \psi>0.$$

To study the increasing and decreasing properties for the AEW function, consider

$$\frac{\partial}{\partial\psi}\frac{K_B(\psi)}{K_A(\psi)}=\frac{K_A(\psi)K_B{}'(\psi)-K_A{}'(\psi)K_B(\psi)}{K_A(\psi)^2},$$

which has the same sign as $\omega_B(\psi)-\omega_A(\psi)$, where

$$\omega_{\square}(\psi)=\frac{K_{\square}{}'(\psi)}{K_{\square}(\psi)}, \quad \square\in\{A,B\}.$$

Letting

$$v(s)=\frac{\partial}{\partial\psi}G_A(\psi)\times\int_0^s\lambda_{y+u}\,du, \quad s>0,$$

then we can write

$$\omega_{\square}(\psi)=\int_0^\infty f_{\square}(s)v(s)ds=\ \mathbb{E}[v(S_{\square})],$$

where $S_{\square}$ denotes the random variable associated with PDF:

$$f_{\;}(s)=\frac{exp(-\int_0^s(\beta+G_{\;}(\psi)\lambda_{y+u})du)}{\int_0^\infty exp\{-\int_0^s\left(\beta+G_{\;}(\psi)\lambda_{y+u}\right)du\}ds}.$$

Note that

$$\frac{f_A(s)}{f_B(s)}=\frac{\int_0^\infty \exp\left(-\int_0^s(\beta+G_B(\psi)\lambda_{y+u})du\right)ds}{\int_0^\infty \exp\left(-\int_0^s(\beta+G_A(\psi)\lambda_{y+u})du\right)ds}\times\exp\left\{\int_0^s[G_B(\psi)-G_A(\psi)]\lambda_{y+u}\,du\right\}$$

which is decreasing in $s>0$ if $\phi<1$, or increasing otherwise. This implies that S_B stochastically dominates (of the first order) S_A if $\phi<1$, and vice versa. Note that $v(s)$ is decreasing in $s>0$ if $\phi<1$, or increasing otherwise. Collectively, we can conclude that $\mathbb{E}[v(S_A)]>\mathbb{E}[v(S_B)]$ for all $\phi>0$, so $K_B(\psi)/K_A(\psi)$ is increasing in ψ. Finally, we can conclude that $\mathrm{AEW}(\psi)$ is increasing in ψ for $\phi<1$, or decreasing otherwise.

The proof is now finished. □

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