Stochastic Analysis of Life Insurance Surplus

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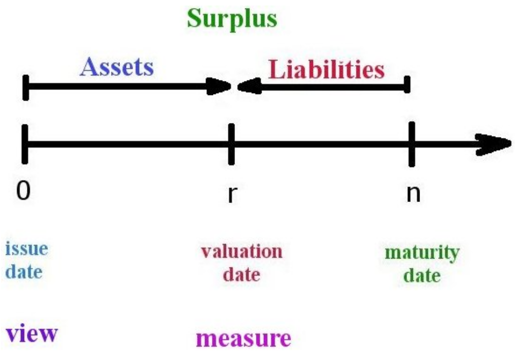
Outline

Motivations

- How risky is the portfolio of life policies?
- How likely is the insurance company to become insolvent in any given year?
- Are premiums and level of initial surplus adequate to ensure high probability of solvency?

Framework



Surplus
Assets
Liabilities
0
r
n
issue date
valuation date
maturity date
view
measure

Risks Facing Insurance Industry

- Mortality
- Investment

Risks Facing Insurance Industry

- Mortality
- Investment
- Expenses
- Persistency
- Other

Decrements due to Mortality: Model

- K_x: *curtate-future-lifetime* of a person aged x
  - number of complete years remaining until death

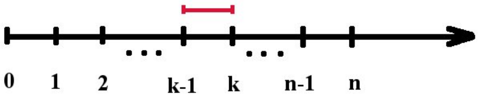
Notation:

- $\mathbf{P}(K_x = k) = {}_{k|}q_x \,, \qquad k = 0, 1, 2, \ldots$
- $\mathbf{P}(K_x > n) = {}_np_x$

- Nonparametric life table
  - Canada 1991, Age Nearest Birthday, Male, Aggregate, Population

Stochastic Rates of Return: Notation

- $\delta(k)$: force of interest in period $(k-1, k]$, $k = 1, 2, \ldots, n$



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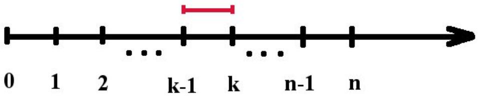

  - δ_k: realization of $\delta(k)$

- $I(s, r)$: force of interest accumulation function

$$I(s,r) = \begin{cases} \sum_{j=s+1}^{r} \delta(j) & \text{if } s < r, \\ 0 & \text{if } s = r. \end{cases}$$

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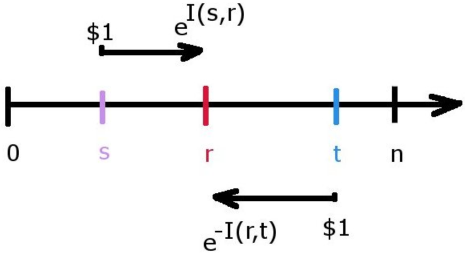

Stochastic Rates of Return: Model

AR(1) model

$$\delta(k) - \delta = \phi\,[\delta(k-1) - \delta] + \varepsilon(k),$$

where

- $\varepsilon(k) \sim N(0,\, \sigma^2)$
- δ: *long-term mean of the process*
- $|\phi| < 1$ *(stationarity)*
- *conditional on starting value* $\delta(0) = \delta_0$

More Assumptions...

Assumptions

- *Future lifetimes are i.i.d.*
- *Lifetimes are independent of rates of return*
- *Identical contracts (i.e., homogeneous portfolio)*

Notation: Life Insurance Policy

- n: term of contract
- x: age at issue
- b: death benefit
  - payable at the end of the year of death
- c: pure endowment benefit
  - payable upon survival to time n
- π: premium
  - payable at the beginning of each year

Notation: Homogeneous Portfolio

$$\mathscr{L}_{i,j}(x) = \begin{cases} 1 & \text{if policyholder } i \text{ aged } x \text{ survives for } j \text{ years,} \\ 0 & \text{otherwise} \end{cases}$$

- $\mathscr{L}_j(x) = \sum_{i=1}^{m} \mathscr{L}_{i,j}(x) \sim \mathrm{BIN}(m, {}_jp_x)$
  - number of policies in force at time j

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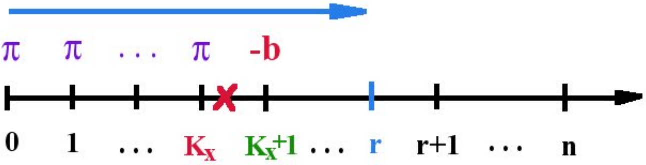
$$\mathscr{D}_{i,j}(x) = \begin{cases} 1 & \text{if policyholder } i \text{ aged } x \text{ dies in year j,} \\ 0 & \text{otherwise} \end{cases}$$

- $\mathscr{D}_j(x) = \sum_{i=1}^{m} \mathscr{D}_{i,j}(x) \sim \text{BIN}(m, {}_{j-1|}q_x)$
  - number of deaths in year j, $j \geq 1$

Retrospective Gain



Retrospective Gain

Definition

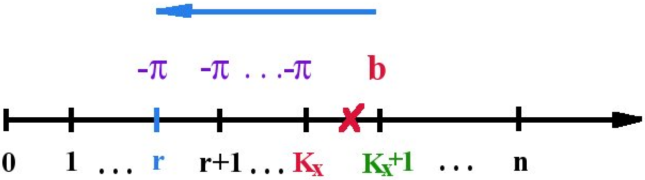
$$RG_r = \sum_{j=0}^{r} RC_j^r \cdot e^{I(j,r)}$$

where

$\mathbf{RC_j^r}$: net cash flow at time j prior to time r, $0 \leq j \leq r$

$$RC_j^r = \pi \cdot \mathscr{L}_j(x) \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathscr{D}_j(x) \cdot \mathbf{1}_{\{j>0\}}$$

Prospective Loss



Prospective Loss

Definition

$$PL_r = \sum_{j=0}^{n-r} PC_j^r \cdot e^{-I(r,r+j)}$$

where

$\mathbf{PC_j^r}$: net cash flow j time units after time r, $0 \leq j \leq n-r$

$$\begin{aligned} PC_j^r = b \cdot \mathscr{D}_{r+j}(x) \cdot \mathbf{1}_{\{j>0\}} \quad &+ \quad c \cdot \mathscr{L}_n(x) \cdot \mathbf{1}_{\{j=n-r\}} \\ &- \quad \pi \cdot \mathscr{L}_{r+j}(x) \cdot \mathbf{1}_{\{j<n-r\}} \end{aligned}$$

Surplus

- Stochastic Surplus

$$S_r^{stoch} = RG_r - PL_r$$

Surplus

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- Accounting Surplus

$$S_r^{acct} = RG_r - {}_rV$$

where ${}_rV \equiv {}_rV(\mathscr{L}_r, \delta(r))$ is the actuarial reserve at time r

  - different ways to calculate ${}_rV$
  - ${}_rV = \mathrm{E}[PL_r | \mathscr{L}_r, \delta(r)]$

Towards Distribution Function . . .

Recall: $S_r^{acct} = RG_r - {}_rV\big(\mathscr{L}_r,\, \delta(r)\big)$

Observation

- *Given values of* $\mathscr{L}_r$ *and* $\delta(r)$, ${}_rV\big(\mathscr{L}_r,\, \delta(r)\big)$ *is constant*

$\Rightarrow$ *cdf of* S_r^{acct} *can be obtained from cdf* RG_r *via*

$$\begin{aligned} & \mathbf{P}[S_r^{acct} \le \xi \,|\, \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] = \\ = \; & \mathbf{P}[RG_r \le \xi + {}_rV(m_r,\, \delta_r) \,|\, \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \end{aligned}$$

Distribution Function: Recursive Approach

- Let $G_t = \sum_{j=0}^{t} RC_j^r \cdot e^{I(j,t)}$, $0 \le t \le r$
- Note: $G_r = RG_r$
- It can be shown: $G_t = G_{t-1} \cdot e^{\delta(t)} + RC_t^r$

Consider a function $g_t(\lambda, m_t, \delta_t)$ given by

$$g_t(\lambda, m_t, \delta_t) = \mathbf{P}[G_t \le \lambda \,|\, \mathscr{L}_t = m_t, \delta(t) = \delta_t] \times \mathbf{P}[\mathscr{L}_t = m_t] \times \times f_{\delta(t)}(\delta_t)$$

Recursive Formula for $g_t(\lambda,\, m_t,\, \delta_t)$

Result

For $1 < t \le r$,

$$g_t(\lambda,\, m_t,\, \delta_t) =$$

$$= \int_{-\infty}^{\infty} \left(\sum_{m_{t-1}=m_t}^{m} \mathbf{P}[\mathscr{L}_t = m_t \,|\, \mathscr{L}_{t-1} = m_{t-1}] \cdot g_{t-1}\left(\frac{\lambda - \eta_t}{e^{\delta_t}},\, m_{t-1},\, \delta_{t-1} \right) \right) \times$$
$$\times\, f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1})\, d\delta_{t-1}$$

where η_t *is the realization of* RC_t^r *for given values of* m_{t-1} *and* m_t*,*

$$\eta_t = \begin{cases} \pi \cdot m_t - b \cdot (m_{t-1} - m_t), & 1 \le t \le r-1, \\ -b \cdot (m_{t-1} - m_t), & t = r \end{cases}$$

with the starting value given by

$$g_1(\lambda,\, m_1,\, \delta_1) = \begin{cases} \mathbf{P}[\mathscr{L}_1(x) = m_1] \cdot f_{\delta(1)}(\delta_1) & \textit{if } G_1 \le \lambda \\ 0 & \textit{otherwise.} \end{cases}$$

Distribution Function of Accounting Surplus

It is easy to show:

Result

$$\mathbf{P}[S_r^{acct} \leq \xi] = \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} g_r(\xi + {}_rV(m_r,\, \delta_r),\, m_r,\, \delta_r)\, d\delta_r$$

Figure: CDF of Accounting Surplus (1000 5-yr Temporary Policies)

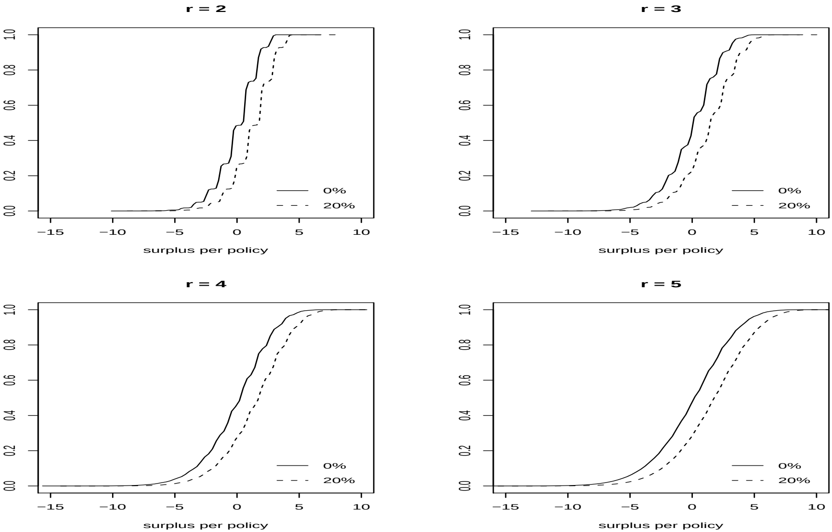
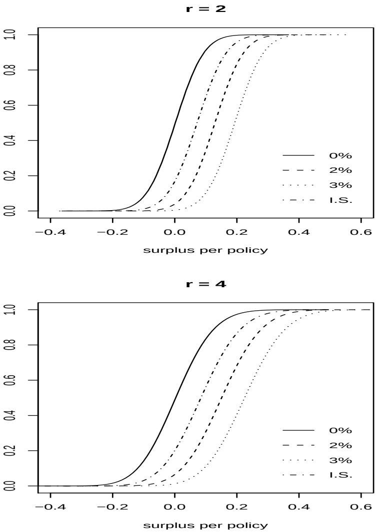


Figure: CDF of Accounting Surplus (Limiting Portfolio)



Future Work

- Under the same assumptions ...
  - Probability of solvency over all years
  - Distribution of stochastic surplus

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- Under the same assumptions ...
  - Probability of solvency over all years
  - Distribution of stochastic surplus
- Extend model to ...
  - general portfolio
  - include expenses, lapses