Some Remarks on *Derivatives Markets*

by

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The parameter δ in the Black-Scholes formula

The Black-Scholes option-pricing formula is given in Chapter 12 of McDonald (2006) without proof. It is important to understand the meaning of each of its parameters. The meaning of the *dividend yield*, δ, is not very clear. The assumption on dividend payments is given in the first sentence on page 132: dividends are paid continuously at a rate that is proportional to the stock price. More precisely, for each share of the stock, the amount of dividends paid between time t and $t+\mathrm{d}t$ is assumed to be $S(t)\delta\mathrm{d}t$. Here, $S(t)$ denotes the price of one share of the stock at time t, $t \geq 0$. (Note that the book also writes $S(t)$ as S_t. The symbol S in formula (12.1) is the same as $S(0)$ and S_0.) This is not exactly a reasonable assumption for stock dividends, but it is needed to obtain formula (12.1). On the other hand, in the context of the Garman-Kohlhagen model for pricing options on currencies (page 381), where $S(t)$ and δ stand for the exchange rate and foreign currency interest rate, respectively, the assumption is reasonable.

It is indicated on page 132 that, if all dividends are re-invested immediately, then one share of the stock at time 0 will grow to $e^{\delta t}$ shares at time t, $t \geq 0$. A calculus proof of this fact is as follows. Let $n(t)$ denote the number of shares of the stock at time t under this immediate reinvestment policy. Thus, $n(0) = 1$. Because the additional number of shares purchased between time t and $t+\mathrm{d}t$ is $\mathrm{d}n(t)$, we have

$$n(t)S(t)\delta\mathrm{d}t = S(t)\mathrm{d}n(t),$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} n(t) = n(t)\delta.$$

Rewriting the last equation as

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln[n(t)] = \delta,$$

integrating both sides, and applying the condition $n(0) = 1$, we obtain the result

$$n(t) = e^{\delta t}.$$

Thus, if we want one share of the stock at time T, we can buy $e^{-\delta T}$ share at time 0 and reinvest all dividends between time 0 and time T. This gives a meaning to the quantity $Se^{-\delta T}$ in formula (12.1). More generally, if we buy $e^{-\delta(T-t)}$ share at time t, $t < T$, and reinvest all dividends between time t and time T, we get one share of the stock at time T. Hence,

$$e^{-\delta(T-t)}S(t) = F_{t,T}^{P}(S), \tag{1}$$

the time-t *prepaid forward price* for delivery of one share of the stock at time T. With $t = 0$, this is formula (5.4) on page 132.

The parameter σ in the Black-Scholes formula

The symbol σ in (12.1) is usually called *volatility*. In finance literature, the term "volatility" does not always have the same meaning. The quantity σ enters the Black-Scholes model via the stochastic differential equation (20.1)

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t). \qquad (2)$$

Because $dS(t)$ is the instantaneous change in the stock price, the left-hand side of the equation, $dS(t)/S(t)$, is the *instantaneous rate of return* due to changes in the stock price. As the first quantity on the right-hand side, αdt, is deterministic, the variance of $dS(t)/S(t)$ is the variance of $\sigma dZ(t)$, which is σ^2 times the variance of $dZ(t)$. As pointed out on page 650, $dZ(t)$ is a normal random variable with variance dt. Thus, the variance of the instantaneous rate of return, $dS(t)/S(t)$, is

$$\sigma^2\, dt,$$

which gives an interpretation for σ.

The stochastic differential equation (20.1) has an explicit solution,

$$S(t) = S(0)\exp[(\alpha - \frac{\sigma^2}{2})t + \sigma Z(t)]; \qquad (3)$$

see (20.13) and (20.29). By means of Itô's Lemma, we can check that the stochastic differential equation (20.1) is indeed satisfied. The exponent, $(\alpha - \frac{\sigma^2}{2})t + \sigma Z(t)$, is the *continuously compounded return* from time 0 to time t (as defined on page 353) due to stock price changes; its variance is

$$\mathrm{Var}[\sigma Z(t)] = \sigma^2 \mathrm{Var}[Z(t)] = \sigma^2 t.$$

In other words, $\sigma\sqrt{t}$ is the standard deviation of the continuously compounded return over the time interval $[0, t]$ (due to changes in the stock price). On page 919 of McDonald (2006), volatility is defined as "[t]he standard deviation of the continuously compounded return on an asset." This is not quite correct, because it has not specified that the length of the time interval is 1.

The *total return* on a stock has two components: return from capital gains (or losses) and return from dividends. The continuous dividend assumption means that the instantaneous rate of return from dividends is the constant δ.

The prepaid forward price version of the Black-Scholes formula

It is pointed out on page 380, after formula (12.5), that "[t]his version of the [Black-Scholes] formula is interesting because the dividend yield and the interest rate do not appear explicitly; they are implicitly incorporated into the prepaid forward prices." In the second half of page 380, formula (12.5) is used to price options on a stock which pays discrete dividends. In this case, the stock price process, $\{S(t)\}$, cannot be a geometric Brownian motion, because there must be a downward jump in the stock price immediately after each dividend is paid. In particular, the logarithm of the stock price cannot be a stochastic process with a constant standard deviation per unit time. So, what is σ in formula (12.5)? It turns out that formula (12.5) follows from the assumption that the stochastic process of the prepaid forward price for delivery of one share of the stock at time T,

$$\{ F_{t,T}^{P}(S)\,;\ 0 \le t \le T\},$$

is a geometric Brownian motion, with σ being the standard deviation per unit time of its natural logarithm.

If the stock pays dividends continuously at a rate proportional to its price, then formula (1) holds. In this case, the prepaid forward price process, $\{ F_{t,T}^{P}(S) \}$, is a geometric Brownian motion if and only if the stock price process, $\{S(t)\}$, is a geometric Brownian motion; both stochastic processes have the same parameter σ. Formula (12.1) is a consequence of (12.5), but the converse is not true because (12.1) is not applicable for pricing options on stocks with discrete dividends.

European Exchange Options

It is pointed out on page 460 that ordinary calls and puts are special cases of exchange options. As hinted in the footnote on page 380, formula (12.5) can be generalized to price European exchange options. For $j = 1, 2$, let $S_j(t)$ denote the price of asset j at time t, $t \ge 0$. Consider a European exchange option whose payoff at time T is

$$\max(S_1(T) - S_2(T), 0).$$

If $\{\ln[F_{\tau,T}^{P}(S_1)];\ 0 \le \tau \le T\}$ and $\{\ln[F_{\tau,T}^{P}(S_2)];\ 0 \le \tau \le T\}$ are a pair of correlated Brownian motions (see page 657), then it can be shown that the time-t price of the European exchange option is

$$F_{t,T}^{P}(S_1)\ N\left(\frac{\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right)$$
$$-\ F_{t,T}^{P}(S_2)\ N\left(\frac{\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)]}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right), \quad 0 \le t < T. \qquad (4)$$

Here, $\mathrm{Var}(\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)]) = \sigma^2 t,\ 0 \le t \le T.$

To emphasize the simplicity of formula (4), let us write $\nu = \sigma\sqrt{T-t}$, and $F_j = F_{t,T}^{P}(S_j)$, $j = 1, 2$. Then, (4) becomes

$$F_1 \times N\left(\frac{\ln[F_1/F_2]}{\nu} + \frac{\nu}{2}\right)\ -\ F_2 \times N\left(\frac{\ln[F_1/F_2]}{\nu} - \frac{\nu}{2}\right),$$

which is not a difficult formula to remember.

To see that formula (14.16) follows from formula (4), we note the assumptions for (14.16): for $j = 1, 2$, $\{S_j(t)\}$ is a geometric Brown motion with volatility σ_j, dividends of amount $S_j(t)\delta_j dt$ are paid between time t and time $t+dt$, and the correlation coefficient between the continuously compounded returns, $\ln[S_1(t)/S_1(0)]$ and $\ln[S_2(t)/S_2(0)]$, is ρ. Thus, similar to (1),

$$F_{t,T}^{P}(S_j) \;=\; e^{-\delta_j(T-t)} S_j(t), \qquad j = 1, 2, \tag{5}$$

and

$$\begin{aligned}\sigma^2 t \;&=\; \mathrm{Var}(\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)])\\ &=\; \mathrm{Var}(\ln[S_1(t)/S_2(t)]) \qquad \text{because of (5)}\\ &=\; \mathrm{Var}(\ln[S_1(t)] \;-\; \ln[S_2(t)])\\ &=\; \mathrm{Var}(\ln[S_1(t)]) \;+\; \mathrm{Var}(\ln[S_2(t)]) \;-\; 2\mathrm{Cov}(\ln[S_1(t)], \ln[S_2(t)])\\ &=\; \sigma_1^2 t \;+\; \sigma_2^2 t \;-\; 2\rho\sigma_1\sigma_2 t,\end{aligned}$$

which is equivalent to (14.17) on page 460.

Black's formula for pricing options on zero-coupon bonds

With the exchange option formula (4), one can derive formula (24.32), which is Black's formula for pricing options on zero-coupon bonds. For $t < T$, consider

$$S_1(t) \;=\; P(t, T + s)$$

and

$$S_2(t) \;=\; K \times P(t, T).$$

Because zero-coupon bonds do not pay dividends, we have

$$F_{t,T}^{P}(S_1) \;=\; S_1(t) \;=\; P(t, T + s)$$

and

$$F_{t,T}^{P}(S_2) \;=\; S_2(t) \;=\; K \times P(t, T).$$

Then,

$$F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2) \;=\; P(t, T + s)/[KP(t, T)].$$

Note that $P(t, T + s)/P(t, T)$ is the time-t forward price for time-T delivery of a zero-coupon bond that pays 1 at time $T + s$. If we can assume that the zero-coupon bond forward price process,

$$\{P(t, T + s)/P(t, T);\;\; 0 \le t \le T\},$$

is a geometric Brownian motion with

$$\mathrm{Var}(\ln[P(t, T + s)/P(t, T)]) \;=\; \sigma^2 t, \qquad 0 \le t \le T$$

(the assumption stated in the first sentence on page 791 is that "the bond forward price is lognormally distributed with constant volatility σ"), then the time-0 price of the European call option with time-T payoff $[P(T, T + s) – K]_+$ is given by formula (4) with $t = 0$. That is, the time-0 price of the call option on the zero-coupon bond is

$$\begin{aligned}&P(0, T + s)\, N\!\left(\frac{\ln[P(0,T+s)/(KP(0,T))]}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right)\\ &-\;\; KP(0, T)\, N\!\left(\frac{\ln[P(0,T+s)/(KP(0,T))]}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right),\end{aligned} \tag{6}$$

which is the formula in footnote 3 on page 791 of McDonald (2006).

If we write

$$F = F_{0,T}[P(T, T + s)] = P(0, T + s)/P(0, T),$$

then (6) becomes

$$P(0, T)\Big[F\times N\left(\frac{\ln(F/K)}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}\right) - K\times N\left(\frac{\ln(F/K)}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}\right)\Big], \qquad (7)$$

which is formula (24.32) on page 791. Alternatively, because

$$\frac{F_{t,T}^P(S_1)}{F_{t,T}^P(S_2)} = \frac{F_{t,T}(S_1)}{F_{t,T}(S_2)},$$

we can rewrite formula (4) as

$$F_{t,T}^P(S_1)\ N\left(\frac{\ln[F_{t,T}(S_1)/F_{t,T}(S_2)]}{\sigma\sqrt{T-t}}+\frac{1}{2}\sigma\sqrt{T-t}\right)$$
$$- F_{t,T}^P(S_2)\ N\left(\frac{\ln[F_{t,T}(S_1)/F_{t,T}(S_2)]}{\sigma\sqrt{T-t}}-\frac{1}{2}\sigma\sqrt{T-t}\right), \quad 0 \le t < T. \qquad (8)$$

Then, (7) follows from (8) with $t = 0$.

The parameters δ and σ in the binomial model

The quantities, δ and σ, also appear in Chapters 10 and 11, which are on binomial models. On page 316, δ is called the continuous dividend yield, and on page 321, σ is called the annualized standard deviation of the continuously compounded stock return. Because binomial models are discrete models, it seems strange that these "continuous-time" concepts are involved. The motivation for incorporating δ and σ in binomial models is sketched in Section 11.3. By letting the length of each time period, *h*, tend to zero (and the number of periods tend to infinity), we can obtain the *risk-neutral* geometric Brownian motion for stock price movements with the dividend yield δ and volatility σ. Note that McDonald (2006) has suggested three pairs of formulas for

$$u = e^{\alpha(h)+\sigma\sqrt{h}}$$

and

$$d = e^{\alpha(h)-\sigma\sqrt{h}}.$$

In (10.10), $\alpha(h) = (r - \delta)h$. In (11.18), $\alpha(h) \equiv 0$, which means $u = 1/d$. In (11.19), $\alpha(h) = (r - \delta - ½\sigma^2)h$. They yield the same limit as $h \to 0$.

The usual model in McDonald (2006) is $\alpha(h) = (r - \delta)h$. A binomial tree so constructed is called a "forward tree" (page 322). In this case, the risk-neutral probabilities are

$$p^* = \frac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \frac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{1}{1+e^{\sigma\sqrt{h}}}$$

and

$$1 - p^* = \frac{e^{\sigma\sqrt{h}}}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{-\sigma\sqrt{h}}};$$

because $\sigma > 0$, we have $p^* < ½ < 1 - p^*$.

Greeks

Greeks are partial derivatives of the option price formula. "The actual formulas for the Greeks appear in Appendix 12.B" (McDonald 2006, p. 382). As the author seemed to want to avoid using calculus in the first half of his book, the definitions given on pages 382 and 383 are numerical approximations. We need to be careful about the units in which changes are measured. For example, it is stated on page 383 that "[t]heta (θ) measures the change in the option price when there is a decrease in the time to maturity of 1 day." However, the mathematical definition for theta is the partial derivative of the option price with respect to t. In the Black-Scholes option-pricing formula, the variable t is (usually) in years. Thus, the definition on page 383 differs from the partial-derivative definition by a factor of 365.

Interest rate

In McDonald (2006), the interest rate is usually a continuously compounded rate, or in actuarial terminology, a force of interest. One exception is Section 24.5 "The Black-Derman-Toy Model" where r_0, r_u, r_d, r_{uu}, etc. are effective annual interest rates. Another exception is the second half of Section 24.3, which is on pricing interest rate caps and caplets.

Chapters 18 and 21

Chapter 18 "The Lognormal Distribution" is not in the syllabus of Exam MFE/3F, but in the syllabus of Exam C/4. Section 18.4 "Lognormal Probability Calculations" is useful for understanding the Black-Scholes formula.

Chapter 21~~, not in the syllabus,~~ derives the celebrated Black-Scholes partial differential equation. McDonald (2006, p. xxiii) states: "Although the Black-Scholes *formula* is famous, the Black-Scholes differential *equation*, discussed in this chapter, is the more profound result." Chapter 21 can be quite helpful for understanding Chapters 12 and 13. Some will find that page 687 demystifies page 394. Others may learn that equation (21.31),

$$\mathrm{E}_t^*[\mathrm{d}V] = V \times r\mathrm{d}t,$$

is a particularly easy-to-remember derivation of the Black-Scholes equation. The instantaneous change in option price, dV, is calculated using Itô's Lemma, which is in the syllabus of Exam MFE/3F. Taking expectation and canceling dt immediately yields the Black-Scholes partial differential equation.