

Multiperiod Optimal Investment-Consumption Strategies with Mortality Risk and Environment Uncertainty

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ABSTRACT

In this paper we investigate three investment-consumption problems for a risk averse investor: (i) an investment only problem that involves utility from only terminal wealth, (ii) an investment-consumption problem that involves utility from only consumption, and (iii) an extended investment-consumption problem that involves utility from both consumption and terminal wealth. We model these problems under a multiperiod setup that incorporates three types of uncertainties: the economic environment uncertainty, the asset return uncertainty, and the mortality uncertainty. By using dynamic programming, analytical expressions of the optimal investment-consumption strategies to these problems are derived. Some economic implications on these results are also discussed.

Key words: Investment-consumption strategy; multiperiod; Mortality risk; Environment uncertainty; Dynamic programming.

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1 Introduction

The problem of optimal investment-consumption has been an area of active research in the last few decades. Samuelson (1969) considered a discrete-time consumption-investment model with the objective of maximizing the overall expected utility of consumption. Using dynamic stochastic programming approach, he succeeded in obtaining the optimal decision for the consumption-investment model. Merton (1969) (see also Merton (1990)) extended the model of Samuelson (1969) to a continuous-time framework and used stochastic optimal control methodology to obtain the optimal portfolio strategy. In particular he showed that under the assumptions of log-normal stock returns and HARA utility, the optimal proportion invested in the risky asset is constant. More recently, Cheung and Yang (2006) investigated a dynamic consumption-investment problem in a regime-switching environment. In this case, the price process of the risky asset was modeled as a discrete-time regime switching process and it was shown that the optimal trading strategy and the consumption strategy are consistent with our intuition in that investors should put a larger proportion of the wealth in the risky asset and consume less when the underlying Markov chain is in a “better” regime. For recent developments and detailed discussion on this subject, we refer the readers to monographs by Karatzas and Shreve (1998) and Korn (1997).

Among these literatures, the uncertainty due to the economy is most commonly considered. In discrete-time case, the economic uncertainty is usually specified by a set of states, each of which is a description of the economic environment for all dates. See for instance, LeRoy and Werner (2001) and the references therein for details. Another crucial assumption which is commonly made in these setups is that once an economic environment is known, the returns of risky assets in any time period are no longer uncertain. This may contradict to what we observe in practice. In recent years, several models have been proposed for addressing this issue. For example, Cheung and Yang (2006) proposed using the Markovian regime switching model to capture the economic uncertainty. In their model, the underlying economy switches among a finite number of states and the returns of risky assets during a time period, which depend on the economic environment at the beginning of that time period, can still be uncertain. The possible economic states in this setup, however, remain unchanged over times. It may be more realistic to assume that

the economic uncertainty is resolved gradually since more and more information is available as time passes. In this paper, we consider a setup which has the capability of modeling two types of uncertainties associate with the economy. They are the *economic environment uncertainty* and the *asset return uncertainty*. The economic environment uncertainty will be described by an event tree generated by finite number of economic states while the asset return uncertainty will allow for randomness of risky asset returns in any time period under any given economic environment. In other words the return on the assets can be stochastic at any time and for any given economic environment.

Another assumption that is commonly made in literature on optimal investment-consumption problems is the exact duration of the planning horizon such as ten or twenty years; that is, at the moment of making an investment-consumption decision, an investor knows with certainty the time of eventual exit. In practice, however, investors may be forced to exit the market before their planned investment horizon due to a variety of reasons such as financial crisis, fatal illness or death of investors. In these situations the time of exit is no longer certain. Consequently, it is of both practical and theoretical importance to develop a comprehensive theory of optimal investment-consumption decisions under uncertain time horizon as induced by the mortality risk. Research on this subject is very limited, especially in the case of discrete-time. Yaari (1965) studied an optimal consumption problem for an individual with uncertain time of death in a simple setup with a pure deterministic investment environment. Hakansson (1969, 1971) extended this work to a discrete-time case with uncertainty including risky assets. Merton (1971) investigated a dynamic optimal portfolios selection problem for an investor with uncertain time of retirement, defined as the time of the first jump of an independent Poisson process with constant intensity. Karatzas and Wang (2001) studied an optimal dynamic investment problem in the case when the uncertain time horizon is a stopping time of asset price filtration. In this paper, we will study the optimality of the consumption-investment problem that reflects the *mortality uncertainty*. Our model assumes that the investor's random time of exiting the market depends on the economic environment uncertainty and the asset return uncertainty and has a known and deterministic conditional probability distribution.

Under the above mentioned three types of uncertainties (the economic environment uncertainty, the asset return uncertainty, and the mortality uncertainty) and

in a multiperiod setup with CRRA preferences, this paper investigates three related optimal consumption-investment problems: (i) an investment only problem that involves utility from only terminal wealth, (ii) an investment-consumption problem that involves utility from only consumption, and (iii) an extended investment-consumption problem that involves utility from both consumption and terminal wealth. We model these problems as optimization problems. By using dynamic programming to tackle the first two problems, analytical expressions of their optimal investment or/and consumption strategies are derived. The results indicates that the optimal investment strategy in the investment only problem is the same as in the investment-consumption problem; that is, the optimal investment and consumption strategy can be separated. Finally, we also demonstrate that the optimal solutions from the investment only problem and the investment-consumption problem can be used to deduce the optimal investment-consumption strategy of the extended investment-consumption problem.

The organization of the paper is as follows. Next section describes the nature of the uncertainties that will be considered in the paper as well as introducing the necessary notations. Sections 3 to 5, respectively, consider the investment only problem, the investment-consumption problem, and the extended investment-consumption problem. Last section concludes the paper.

2 Uncertainties and Notations

We consider an investor who wants to make a multiperiod consumption-investment decision. The investor enters the market at time 0 with initial endowment of $W_0 > 0$. We assume that the investor has a planned investment horizon T , where T is a fixed integer and can be interpreted as the remaining time of retirement of the investor. By partitioning the time horizon into T time periods, as indexed by $t = 0, 1, \dots, T$, the investor, at the beginning of each such time period, can distribute his wealth among consumption and investment. Here and thereafter the t -th time period refers to the time interval $[t, t + 1)$. If the investor were to invest, then he needs to decide the appropriate allocation among $J + 1$ assets, which are indexed by $j = 0, 1, \dots, J$.

The investor is assumed to face three kinds of uncertainty: the uncertainty of economic environment, the uncertainty of the returns of assets, and the uncertainty

of the time of death of the investor. We now describe each of these uncertainties in greater details.

With regards to the uncertainty induced by the economic environment, we specify it by a set of finite number of states, Ω , which is equipped with a probability measure. Each of the states represents an economic environment for all times $t = 0, 1, \dots, T$. The information of the investor at time t is described by a σ -field \mathcal{F}_t , which is generated by a partition of Ω .¹ At time $t = 0$, the investor has no information about the state, thus $\mathcal{F}_0 = \{\emptyset, \Omega\}$. At time $t = T$, the investor has full information, and hence $\mathcal{F}_T = \sigma\{\{\omega\} : \omega \in \Omega\}$. At time $t = 1, \dots, T - 1$, the investor has intermediate amount of information. We assume that the partition becomes finer as time increases. Thus the sequence of σ -fields, $\{\mathcal{F}_t\}$, is increasing. In other words, the element of \mathcal{F}_{t+1} to which a state belongs to is a subset of the element of \mathcal{F}_t to which it belongs. The $(T + 1)$ -tuple of σ -fields, $\{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$, simply denoted by \mathcal{F} , is called the information filtration. The corresponding collection of partitions is known as an event tree. Each element of σ -field \mathcal{F}_t is called a time- t event and corresponds to a node of the event tree. An event at a time can be intuitively interpreted as an economic state at that time. Denote by ξ_t a time- t event. The successors of the event ξ_t are the event $\xi_s \subset \xi_t$ for $s > t$. The immediate successors of ξ_t are the events $\xi_{t+1} \subset \xi_t$. The predecessors of ξ_t are the events $\xi_s \supset \xi_t$ for $s < t$. The immediate predecessor of ξ_t is the unique event $\xi_{t-1} \supset \xi_t$, which sometimes is conveniently denoted by ξ_t^- .

For the second uncertainty associated with the uncertainty of the returns of the assets, we assume that the returns of these assets at each time period depend on the economic state at the beginning of that time period. We use r_{j,ξ_t} to denote the random return of asset j in the t -th time period at event ξ_t . The random return r_{j,ξ_t} is assumed to be strictly positive and integrable for any j and ξ_t , $t = 0, \dots, T - 1$. We also assume that the one-period random return vectors of the $J + 1$ assets in different time periods are independent, i.e., $(r_{0,\xi_t}, \dots, r_{J,\xi_t})$, $t = 0, 1, \dots, T - 1$, are independent for any given $\xi_t \in \mathcal{F}_t$, $t = 0, 1, \dots, T - 1$. We further assume that for any $t = 0, 1, \dots, T - 1$ and any $\xi_t \in \mathcal{F}_t$, the random return $(r_{0,\xi_t}, \dots, r_{J,\xi_t})$ in the t -th time period at event ξ_t is independent of the state variable ξ_{t+1} at the beginning of the next time period.

¹A partition of Ω is a collection of subsets of Ω such that each state ω belongs to exactly one element of the partition.

By $W_t(\xi_t)$ we denote the wealth of the investor at time- t event ξ_t . The random variable $W_{t+1}(\xi_{t+1})$ depends on the random return $(r_{0,\xi_t}, \dots, r_{J,\xi_t})$ in the previous time period t at event $\xi_t = \xi_{t+1}^-$.

It would not make sense to consider portfolios, consumptions, and so forth at time t that differs in states that cannot be distinguished based on the information available to the investor at time t . So we assume that the portfolio process and the consumption process are adapted to the filtration generated by \mathcal{F}_t and $\{(r_{0,\xi_s}, \dots, r_{J,\xi_s}) : s \leq t-1\}$. Under this assumption, the percentage of the wealth invested in asset j at time t , denoted by θ_{jt} , is measurable with respect to the σ -field generated by \mathcal{F}_t and $\{(r_{0,\xi_s}, \dots, r_{J,\xi_s}) : s \leq t-1\}$. We denote the common value of θ_{jt} on ξ_t by $\theta_{jt}(\xi_t)$ and simply call it the fraction of wealth invested in asset j at time- t event ξ_t . Now let the column vector $\boldsymbol{\theta}_t(\xi_t) = (\theta_{1t}(\xi_t), \dots, \theta_{Jt}(\xi_t))'$ be the portfolio of assets $1, 2, \dots, J$ at time- t event ξ_t , where the superscript $'$ denotes the transpose of a vector. Then $\theta_{0,t}(\xi_t) = 1 - \mathbf{1}'\boldsymbol{\theta}_t(\xi_t)$, where $\mathbf{1}$ is the J -dimensional vector with all entries equal to 1. To avoid the possibility of the wealth becoming negative, our model does not permit short-selling of any asset; that is, the portfolio weight $\boldsymbol{\theta}_t(\xi_t)$ at any event ξ_t is constrained to lie in the convex set

$$\Theta := \{\boldsymbol{\theta} \in \mathbb{R}^J : 0 \leq \theta_j \leq 1, j = 1, \dots, J; \mathbf{1}'\boldsymbol{\theta} \leq 1\}. \quad (1)$$

We refer such a adapted process $\{\boldsymbol{\theta}_t(\xi_t) \in \Theta : \xi_t \in \mathcal{F}_t, t = 0, 1, \dots, T-1\}$ as an admissible investment strategy.

Similarly, by $c_t(\xi_t)$ we denote the consumption of the investor at event ξ_t . We assume that it can not be negative and can not exceed the wealth at that event, i.e., $c_t(\xi_t) \in [0, W_t(\xi_t)]$. We refer such a adapted process $c := \{c_t(\xi_t) : \xi_t \in \mathcal{F}_t, t = 0, 1, \dots, T-1\}$ as an admissible consumption strategy.

It follows immediately from the above notations that the wealth evolves over time according to

$$\begin{aligned} W_{t+1}(\xi_{t+1}) &= [W_t(\xi_t) - c_t(\xi_t)] \sum_{j=0}^J r_{j,\xi_t} \theta_{jt}(\xi_t) \\ &= [W_t(\xi_t) - c_t(\xi_t)] [r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta}_t(\xi_t)] \end{aligned} \quad (2)$$

for $\xi_t \in \mathcal{F}_t$, $\xi_{t+1} \subset \xi_t$, $t = 0, \dots, T-1$, where $\mathbf{R}_{\xi_t} = (R_{1,\xi_t}, \dots, R_{J,\xi_t})'$ and $R_{j,\xi_t} = r_{j,\xi_t} - r_{0,\xi_t}$. Obviously, the wealth process W is non-negative.

For the last uncertainty due to the contingent time of death of the investor, we assume that if the investor dies during the time period t , then he will exit the market at the end of this time period, or at time $t + 1$. Let τ denote the random time of exit of the investor due to death, which takes integer values $1, 2, \dots$, then the actual time of exit of the investor is $T \wedge \tau := \min\{T, \tau\}$. We assume that the probability distribution of τ conditional on economic environment \mathcal{F} and asset returns \mathcal{R} is known and is denoted by

$$\Pr(\tau = t | \mathcal{F}, \mathcal{R}) = q_t, \quad t = 0, 1, \dots,$$

where $q_0 = 0, q_t \geq 0, t = 0, 1, \dots$, and $\sum_{t=0}^{\infty} q_t = 1$.

3 Investment Only Problem

We begin our analysis by first considering an investment only problem. In this special case, the investor's objective is to maximize the expected utility of his terminal wealth over all admissible investment strategies. Mathematically, this is equivalent to solving the following optimization problem:

$$\max_{\boldsymbol{\theta}_t(\xi_t) \in \Theta; \xi_t \in \mathcal{F}_t, t=0, \dots, T-1} \mathbf{E} [u(W_{T \wedge \tau})], \quad (3)$$

subject to the budget constraint

$$W_{t+1}(\xi_{t+1}) = W_t(\xi_t) [r_{0, \xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta}_t(\xi_t)], \quad (4)$$

where $\xi_t \in \mathcal{F}_t, \xi_{t+1} \subset \xi_t, t = 0, \dots, T - 1$, \mathbf{E} is the expectation operator, and u is an utility function of wealth. In this paper, we assume that the investor is a power utility optimizer so that u has the following representation:

$$u(w) = \frac{1}{\gamma} w^\gamma, \quad (5)$$

where γ is a constant that lies in the interval $(-\infty, 0) \cup (0, 1)$.

Note that the above formulation involves optimizing over an uncertain exit time. If the investor is still alive at time T , then he is maximizing his expected utility of retirement income. If death occurs before the scheduled retired time T , then the investor is maximizing his expected utility of bequest.

3.1 Problem Reformulation and the Auxiliary Function

As noted above, problem (3) deals with uncertain terminal time. To obtain its optimal solution, it is convenient to first transform it into an equivalent optimization problem with certain terminal time, as we demonstrate in this subsection.

Using the iterative expectation formula, we have

$$\begin{aligned} \mathbb{E}[u(W_{T \wedge \tau})] &= \mathbb{E}[\mathbb{E}[u(W_{T \wedge \tau}) | \mathcal{F}, \mathcal{R}]] = \mathbb{E}\left[\sum_{t=1}^{\infty} q_t u(W_{T \wedge t})\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{T-1} q_t u(W_t) + \sum_{t=T}^{\infty} q_t u(W_T)\right] = \mathbb{E}\left[\sum_{t=1}^T p_t u(W_t)\right], \end{aligned}$$

where

$$p_t = \begin{cases} q_t, & t = 0, 1, \dots, T-1 \\ \sum_{s=T}^{\infty} q_s, & t = T. \end{cases}$$

Hence, p_T denotes the probability of the death occurs after time $T-1$ or equivalently the probability of the investor survives after time $T-1$. Consequently, the optimization problem (3) can be reformulated as

$$\max_{\boldsymbol{\theta}_t(\xi_t) \in \Theta; \xi_t \in \mathcal{F}_t, t=0, \dots, T-1} \mathbb{E}\left[\sum_{t=1}^T p_t u(W_t)\right]. \quad (6)$$

To solve the above dynamic optimization problem, it is useful to introduce the following auxiliary function. Let us define, for any $\gamma \in (-\infty, 0) \cup (0, 1)$, $t \in \{0, 1, \dots, T-1\}$ and $\xi_t \in \mathcal{F}_t$, a function $Q_{\xi_t}^{\gamma} : \Theta \rightarrow \mathbb{R}$ such that

$$Q_{\xi_t}^{\gamma}(\boldsymbol{\theta}) = \mathbb{E}\left[\frac{1}{\gamma}(r_{0, \xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta})^{\gamma} \middle| \xi_t\right]. \quad (7)$$

Clearly, $Q_{\xi_t}^{\gamma}(\boldsymbol{\theta})$ can be interpreted as the expected utility of the one-period investment return for an initial investment of \$1 with portfolio weight $\boldsymbol{\theta}$ and given the time- t event ξ_t . Important properties associated with this function are summarized in the following lemma.

Lemma 3.1 *For any fixed $\gamma \in (-\infty, 0) \cup (0, 1)$, $t \in \{0, 1, \dots, T-1\}$ and $\xi_t \in \mathcal{F}_t$, the function $Q_{\xi_t}^{\gamma} : \Theta \rightarrow \mathbb{R}$ is*

- (i) *well-defined, if the additional condition that r_{j, ξ_t}^{γ} is integrable for all j is imposed when $\gamma < 0$;*

(ii) *strictly concave;*

(iii) *continuous.*

Proof (i) We need only to prove that $(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta})^\gamma$ is integrable for any given $\boldsymbol{\theta} \in \Theta$. First we note that

$$\min_{0 \leq j \leq J} r_{j,\xi_t} \leq r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta} \leq \sum_{j=0}^J r_{j,\xi_t}.$$

When $0 < \gamma < 1$, we have

$$(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta})^\gamma \leq \left(\sum_{j=0}^J r_{j,\xi_t} \right)^\gamma \leq \max \left\{ 1, \sum_{j=0}^J r_{j,\xi_t} \right\} \leq 1 + \sum_{j=0}^J r_{j,\xi_t},$$

which is integrable by our assumption that r_{j,ξ_t} is integrable for all j . When $\gamma < 0$, we have

$$(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta})^\gamma \leq \left(\min_{0 \leq j \leq J} r_{j,\xi_t} \right)^\gamma = \max_{0 \leq j \leq J} r_{j,\xi_t}^\gamma \leq \sum_{j=0}^J r_{j,\xi_t}^\gamma,$$

which is integrable under the additional condition that r_{j,ξ_t}^γ is integrable for all j .

(ii) Let $\hat{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}} \in \Theta$, $\hat{\boldsymbol{\theta}} \neq \tilde{\boldsymbol{\theta}}$, and $\lambda \in (0, 1)$. Then

$$\begin{aligned} Q_{\xi_t}^\gamma \left(\lambda \hat{\boldsymbol{\theta}} + (1 - \lambda) \tilde{\boldsymbol{\theta}} \right) &= \mathbb{E} \left[\frac{1}{\gamma} \left(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \left(\lambda \hat{\boldsymbol{\theta}} + (1 - \lambda) \tilde{\boldsymbol{\theta}} \right) \right)^\gamma \middle| \xi_t \right] \\ &= \mathbb{E} \left[\frac{1}{\gamma} \left(\lambda \left(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \hat{\boldsymbol{\theta}} \right) + (1 - \lambda) \frac{1}{\gamma} \left(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \tilde{\boldsymbol{\theta}} \right) \right)^\gamma \middle| \xi_t \right] \\ &> \lambda \mathbb{E} \left[\frac{1}{\gamma} \left(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \hat{\boldsymbol{\theta}} \right)^\gamma \middle| \xi_t \right] + (1 - \lambda) \mathbb{E} \left[\frac{1}{\gamma} \left(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \tilde{\boldsymbol{\theta}} \right)^\gamma \middle| \xi_t \right] \\ &= \lambda Q_{\xi_t}^\gamma \left(\hat{\boldsymbol{\theta}} \right) + (1 - \lambda) Q_{\xi_t}^\gamma \left(\tilde{\boldsymbol{\theta}} \right), \end{aligned}$$

where the inequality follows from the strict concavity of the function x^γ/γ . Hence the function $Q_{\xi_t}^\gamma$ is strictly concave.

(iii) From the proof of the assertion (i), the collection of random variables $\{(r_{0,\xi_t} + \mathbf{R}'_{\xi_t} \boldsymbol{\theta})^\gamma : \boldsymbol{\theta} \in \Theta\}$ is dominated by an integrable random variable. This together with the Dominated Convergence Theorem yields the continuity of the function $Q_{\xi_t}^\gamma$. \square

In what follows, we impose the additional assumption that r_{j,ξ_t}^γ is integrable for all $j = 0, \dots, J$, $t = 0, \dots, T-1$ and $\xi_t \in \mathcal{F}_t$ when $\gamma < 0$. An immediate consequence of Lemma 3.1 is that the function $Q_{\xi_t}^\gamma$ achieves its maximum value on the bounded

closed set Θ at an unique point. We use the notation $\boldsymbol{\theta}_{\xi_t}^{(\gamma)}$ to denote such unique point and use $Q_{\xi_t}^{\gamma(1)}/\gamma$ to denote its corresponding maximum value $Q_{\xi_t}^{\gamma}(\boldsymbol{\theta}_{\xi_t}^{(\gamma)})$. Clearly, $Q_{\xi_t}^{\gamma(1)}$ is non-negative. Furthermore, by setting the initial condition

$$Q_{\xi_t}^{\gamma(0)} = 1, \quad \xi_t \in \mathcal{F}_t, \quad t = 0, 1, \dots, T,$$

we recursively define for $t = 0, 1, \dots, T - 1$,

$$Q_{\xi_t}^{\gamma(k+1)} = Q_{\xi_t}^{\gamma(1)} \mathbb{E} \left[Q_{\xi_{t+1}}^{\gamma(k)} \middle| \xi_t \right], \quad k = 0, 1, \dots, T - t - 1. \quad (8)$$

Note that $Q_{\xi_t}^{\gamma(k)} \geq 0$ for $t = 0, 1, \dots, T - 1$ and $k = 0, 1, \dots, T - t - 1$.

3.2 Optimal Investment Strategy

We are now ready to solve our reformulated optimization problem (6) by using the dynamic programming approach. This entails defining the value function $v_t : \mathbb{R}_+ \times \mathcal{F}_t$ as:

$$v_t(W_t, \xi_t) := \max_{\substack{\boldsymbol{\theta}_s(\xi_s) \in \Theta; \\ s=t, \dots, T-1}} \mathbb{E} \left[\sum_{s=t}^T p_s u(W_s) \middle| \xi_t \right],$$

for $t \in \{0, 1, \dots, T - 1\}$ with the utility function u given by (5).

Our objective is to compute the optimal expected utility $v_0(W_0, \xi_0)$ and the corresponding optimal investment strategy. By the Bellman optimality principle of dynamic programming, we have the following recursive equation for the value functions:

$$v_t(W_t, \xi_t) = p_t u(W_t) + \max_{\boldsymbol{\theta}_t(\xi_t) \in \Theta} \mathbb{E} [v_{t+1}(W_{t+1}, \xi_{t+1}) | \xi_t], \quad t = T - 1, \dots, 0, \quad (9)$$

where $\xi_{t+1} \subset \xi_t$ and W_{t+1} is given by (4); together with the terminal condition

$$v_T(W_T, \xi_T) = p_T u(W_T).$$

Theorem 3.1 *For investment only problem (3), the value functions can be represented as*

$$v_t(W_t, \xi_t) = \frac{W_t^\gamma}{\gamma} \sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-t-n)}, \quad t = 0, 1, \dots, T, \quad (10)$$

and the optimal investment strategy is given by

$$\boldsymbol{\theta}_t(\xi_t) = \boldsymbol{\theta}_{\xi_t}^{(\gamma)}, \quad t = 0, 1, \dots, T - 1. \quad (11)$$

Proof We prove the theorem by induction. It is obvious that expressions (10) and (11) are true for $t = T$. For $t = T - 1$, we have (for brevity, here $\boldsymbol{\theta}_{T-1}(\xi_{T-1})$ is denoted by $\boldsymbol{\theta}$)

$$\begin{aligned}
v_{T-1}(W_{T-1}, \xi_{T-1}) &= \max_{\boldsymbol{\theta} \in \Theta} \{p_{T-1}u(W_{T-1}) + \mathbb{E}[v_T(W_T, \xi_T)|\xi_{T-1}]\} \\
&= p_{T-1}u(W_{T-1}) + \max_{\boldsymbol{\theta} \in \Theta} \mathbb{E}[p_T u(W_T)|\xi_{T-1}] \\
&= \frac{1}{\gamma} p_{T-1} W_{T-1}^\gamma + p_T W_{T-1}^\gamma \max_{\boldsymbol{\theta} \in \Theta} \mathbb{E} \left[\frac{1}{\gamma} (r_{0, \xi_{T-1}} + \mathbf{R}'_{\xi_{T-1}} \boldsymbol{\theta})^\gamma \middle| \xi_{T-1} \right] \\
&= \frac{1}{\gamma} p_{T-1} W_{T-1}^\gamma + p_T W_{T-1}^\gamma \max_{\boldsymbol{\theta} \in \Theta} Q_{\xi_{T-1}}^\gamma(\boldsymbol{\theta}) \\
&= \frac{1}{\gamma} p_{T-1} W_{T-1}^\gamma + p_T W_{T-1}^\gamma Q_{\xi_{T-1}}^\gamma(\boldsymbol{\theta}_{\xi_{T-1}}^{(\gamma)}) \\
&= \frac{1}{\gamma} p_{T-1} W_{T-1}^\gamma + \frac{1}{\gamma} p_T W_{T-1}^\gamma Q_{\xi_{T-1}}^{\gamma(1)} \\
&= \frac{1}{\gamma} W_{T-1}^\gamma \left(p_{T-1} Q_{\xi_{T-1}}^{\gamma(0)} + p_T Q_{\xi_{T-1}}^{\gamma(1)} \right),
\end{aligned}$$

which shows that (10) and (11) are true for $t = T - 1$. Now assuming both (10) and (11) hold for t , then for $t - 1$ we have (again $\boldsymbol{\theta}_{t-1}(\xi_{t-1})$ is denoted by $\boldsymbol{\theta}$)

$$\begin{aligned}
v_{t-1}(W_{t-1}, \xi_{t-1}) &= \max_{\boldsymbol{\theta} \in \Theta} \{p_{t-1}u(W_{t-1}) + \mathbb{E}[v_t(W_t, \xi_t)|\xi_{t-1}]\} \\
&= \frac{1}{\gamma} p_{t-1} W_{t-1}^\gamma + \max_{\boldsymbol{\theta} \in \Theta} \mathbb{E} \left[\frac{W_t^\gamma}{\gamma} \sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-t-n)} \middle| \xi_{t-1} \right] \\
&= \frac{1}{\gamma} p_{t-1} W_{t-1}^\gamma + W_{t-1}^\gamma \mathbb{E} \left[\sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-t-n)} \middle| \xi_{t-1} \right] \max_{\boldsymbol{\theta} \in \Theta} \mathbb{E} \left[\frac{1}{\gamma} (r_{0, \xi_{t-1}} + \mathbf{R}'_{\xi_{t-1}} \boldsymbol{\theta})^\gamma \middle| \xi_{t-1} \right] \\
&= \frac{1}{\gamma} p_{t-1} W_{t-1}^\gamma + W_{t-1}^\gamma \mathbb{E} \left[\sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-t-n)} \middle| \xi_{t-1} \right] \max_{\boldsymbol{\theta} \in \Theta} Q_{\xi_{t-1}}^\gamma(\boldsymbol{\theta}) \\
&= \frac{1}{\gamma} p_{t-1} W_{t-1}^\gamma + W_{t-1}^\gamma \mathbb{E} \left[\sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-t-n)} \middle| \xi_{t-1} \right] Q_{\xi_{t-1}}^\gamma(\boldsymbol{\theta}_{\xi_{t-1}}^{(\gamma)}) \\
&= \frac{1}{\gamma} p_{t-1} W_{t-1}^\gamma + \frac{1}{\gamma} W_{t-1}^\gamma \mathbb{E} \left[\sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-t-n)} \middle| \xi_{t-1} \right] Q_{\xi_{t-1}}^{\gamma(1)} \\
&= \frac{1}{\gamma} W_{t-1}^\gamma \left[p_{t-1} + \sum_{n=0}^{T-t} p_{T-n} Q_{\xi_{t-1}}^{\gamma(1)} \mathbb{E} \left[Q_{\xi_t}^{\gamma(T-t-n)} \middle| \xi_{t-1} \right] \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\gamma} W_{t-1}^\gamma \left[p_{t-1} + \sum_{n=0}^{T-t} p_{T-n} Q_{\xi_t}^{\gamma(T-(t-1)-n)} \right] \\
&= \frac{1}{\gamma} W_{t-1}^\gamma \sum_{n=0}^{T-(t-1)} p_{T-n} Q_{\xi_t}^{\gamma(T-(t-1)-n)},
\end{aligned}$$

which shows that (10) and (11) are also true for $t - 1$. This completes the induction proof. \square

It is of interest to note that while our investment only problem (3) explicitly reflects the possibility of early exit due to the death of the investor, its optimal investment strategy is independent of the mortality risk and the planned investment horizon, as assured by Theorem 3.1. In other words, the investment strategies optimally adopted by the investor are exactly the same regardless of whether he takes into consideration the mortality risk. It should, however, be emphasized that the optimal investment strategy depends on both the economic environments and the asset returns. More specifically, the optimal proportion of the wealth invested in any asset at any time depends on the event (economic state) at that time and on the asset returns in the following time period, but not on the wealth at that time nor on the remaining time of investment horizon. In addition, the optimal expected utility $v_0(W_0, \xi_0)$ of the investor increases with the planned time horizon T as well as with the initial wealth W_0 , which are consistent with our intuition.

4 Investment-Consumption Problem

In this section, we focus on the problem of optimal investment and consumption for an investor. The investor needs to decide an appropriate wealth allocation among assets and consumption at the beginning of each time period so as to maximize the expected utility from consumption over all times before exiting. In other words, the problem of the investor can be formulated as

$$\max_{\substack{\theta_t(\xi_t) \in \Theta, c_t(\xi_t) \in [0, W_t(\xi_t)]; \\ \xi_t \in \mathcal{F}_t, t=0, \dots, T-1}} \mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \right], \quad (12)$$

subject to the budget constraint (2), where U is an utility function of consumption which again is assumed to be the power utility but with parameter μ ; i.e.

$$U(x) = \frac{1}{\mu} x^\mu, \quad (13)$$

where $\mu \in (0, 1)$ is a given constant.

4.1 Problem Reformulation and the Auxiliary Function

Note again that the investment-consumption problem (12) maximizes the expected utility over an uncertain exit time. This problem can be converted into an equivalent optimization problem with certain terminal time by first recognizing that

$$\begin{aligned}
\mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \right] &= \mathbb{E} \left[\mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \middle| \mathcal{F} \right] \right] \\
&= \mathbb{E} \left[\sum_{s=1}^{\infty} q_s \sum_{t=0}^{T \wedge s - 1} U(c_t) \right] \\
&= \mathbb{E} \left[\sum_{s=1}^T q_s \sum_{t=0}^{s-1} U(c_t) + \sum_{s=T+1}^{\infty} q_s \sum_{t=0}^{T-1} U(c_t) \right] \\
&= \mathbb{E} \left[\sum_{t=0}^{T-1} \sum_{s=t+1}^T q_s U(c_t) + \sum_{t=0}^{T-1} \sum_{s=T+1}^{\infty} q_s U(c_t) \right] \\
&= \mathbb{E} \left[\sum_{t=0}^{T-1} y_t U(c_t) \right],
\end{aligned}$$

where $y_t := \sum_{s=t+1}^{\infty} q_s$ captures the probability that the investor survives until (or dies after) time t . Then the optimization problem with certain terminal time that is equivalent to (12) is

$$\max_{\substack{\theta_t(\xi_t) \in \Theta, c_t(\xi_t) \in [0, W_t(\xi_t)]; \\ \xi_t \in \mathcal{F}_t, t=0, \dots, T-1}} \mathbb{E} \left[\sum_{t=0}^{T-1} y_t U(c_t) \right]. \quad (14)$$

The following auxiliary result, which is from Cheung and Yang (2006), is useful in solving the above optimization problem.

Lemma 4.1 *Suppose that $\lambda > 0, w > 0$, and $0 < \mu < 1$ are fixed constants. The function $f : [0, w] \rightarrow \mathbb{R}$ defined by*

$$f(c) = c^\mu + \lambda(w - c)^\mu$$

achieves its unique maximum

$$f(c^*) = w^\mu \left(1 + \lambda^{\frac{1}{1-\mu}} \right)^{1-\mu}$$

at

$$c^* = \frac{w}{1 + \lambda^{\frac{1}{1-\mu}}}.$$

4.2 Optimal Investment-Consumption Strategy

We now turn to solving the optimization problem (14). For $t \in \{0, 1, \dots, T-1\}$, let us define the value function $V_t : \mathbb{R}_+ \times \mathcal{F}_t$ as

$$V_t(W_t, \xi_t) := \max_{\substack{\boldsymbol{\theta}_s(\xi_s) \in \Theta, c_s(\xi_s) \in [0, W_s(\xi_s)]; \\ \xi_s \subset \xi_t, s=t, \dots, T-1}} \mathbb{E} \left[\sum_{s=t}^{T-1} y_s U(c_s) \middle| \xi_t \right],$$

where $W_t(\xi_t) = W_t$ and U is from (13). The Bellman optimality principle of dynamic programming implies the following recursive equation for the value functions:

$$V_t(W_t, \xi_t) = \max_{\boldsymbol{\theta}_t(\xi_t) \in \Theta, c_t(\xi_t) \in [0, W_t]} \{y_t U(c_t) + \mathbb{E}[V_{t+1}(W_{t+1}, \xi_{t+1}) | \xi_t]\}, \quad t = T-2, \dots, 0, \quad (15)$$

where $\xi_{t+1} \subset \xi_t$ and W_{t+1} is given by (2); together with the terminal condition

$$\begin{aligned} V_{T-1}(W_{T-1}, \xi_{T-1}) &= \max_{\boldsymbol{\theta}_{T-1}(\xi_{T-1}) \in \Theta, c_{T-1}(\xi_{T-1}) \in [0, W_{T-1}]} y_{T-1} U(c_{T-1}(\xi_{T-1})) \\ &= y_{T-1} U(W_{T-1}). \end{aligned} \quad (16)$$

By initializing

$$L_{\xi_{T-1}} := 0 \quad (17)$$

for $\xi_{T-1} \in \mathcal{F}_{T-1}$, we recursively define

$$L_{\xi_t} = \left\{ \frac{y_{t+1}}{y_t} Q_{\xi_t}^{\mu(1)} \mathbb{E} \left[(1 + L_{\xi_{t+1}})^{1-\mu} \middle| \xi_t \right] \right\}^{\frac{1}{1-\mu}} \quad (18)$$

where $\xi_t \in \mathcal{F}_t, t = 0, 1, \dots, T-2$. In the above equation, $Q_{\xi_t}^{\mu(1)}/\mu$ denotes the maximum value of the function $Q_{\xi_t}^{\mu}(\boldsymbol{\theta})$ where $Q_{\xi_t}^{\mu}(\boldsymbol{\theta})$ is similarly defined as in (7) except replacing the parameter γ by μ . Analogously, we also use the notation $\boldsymbol{\theta}_{\xi_t}^{(\mu)}$ to denote the point at which the function $Q_{\xi_t}^{\mu}(\boldsymbol{\theta})$ attains its maximum. Note that L_{ξ_t} is nonnegative for all $\xi_t \in \mathcal{F}_t, t = 0, 1, \dots, T-1$.

We now present the optimal solution to our investment-consumption problem (12), which is summarized in the following theorem:

Theorem 4.1 For investment-consumption problem (12), the value functions are given by

$$V_t(W_t, \xi_t) = y_t \frac{W_t^\mu}{\mu} (1 + L_{\xi_t})^{1-\mu}, \quad \xi_t \in \mathcal{F}_t, \quad t = 0, 1, \dots, T-1, \quad (19)$$

and the optimal investment-consumption strategy is given by

$$\boldsymbol{\theta}_t(\xi_t) = \boldsymbol{\theta}_{\xi_t}^{(\mu)}, \quad (20)$$

$$c_t(\xi_t) = W_t (1 + L_{\xi_t})^{-1} \quad (21)$$

for $\xi_t \in \mathcal{F}_t, t = 0, 1, \dots, T-1$.

Proof The above result can similarly be proved by induction. The conclusions of the theorem are obviously true for $t = T-1$. For $t = T-2$, we have (for brevity, here $c_{T-2}(\xi_{T-2})$ and $\boldsymbol{\theta}_{T-2}(\xi_{T-2})$ are denoted, respectively, by c and $\boldsymbol{\theta}$)

$$\begin{aligned} & V_{T-2}(W_{T-2}, \xi_{T-2}) \\ &= \max_{c \in [0, W_{T-2}], \boldsymbol{\theta} \in \Theta} \{y_{T-2}U(c) + \mathbb{E}[V_{T-1}(W_{T-1}, \xi_{T-1}) | \xi_{T-2}]\} \\ &= \max_{c \in [0, W_{T-2}], \boldsymbol{\theta} \in \Theta} \left\{ y_{T-2}U(c) + \mathbb{E} \left[y_{T-1} \frac{W_{T-1}^\mu}{\mu} \middle| \xi_{T-2} \right] \right\} \\ &= \max_{c \in [0, W_{T-2}], \boldsymbol{\theta} \in \Theta} \left\{ y_{T-2}U(c) + \mathbb{E} \left[y_{T-1} (W_{T-2} - c)^\mu \frac{1}{\mu} (r_{0, \xi_{T-2}} + \mathbf{R}'_{\xi_{T-2}} \boldsymbol{\theta})^\mu \middle| \xi_{T-2} \right] \right\} \\ &= \max_{c \in [0, W_{T-2}], \boldsymbol{\theta} \in \Theta} \left\{ y_{T-2}U(c) + y_{T-1} (W_{T-2} - c)^\mu Q_{\xi_{T-2}}^\mu(\boldsymbol{\theta}) \right\} \\ &= \max_{c \in [0, W_{T-2}]} \left\{ y_{T-2}U(c) + y_{T-1} (W_{T-2} - c)^\mu \max_{\boldsymbol{\theta} \in \Theta} Q_{\xi_{T-2}}^\mu(\boldsymbol{\theta}) \right\} \\ &= \max_{c \in [0, W_{T-2}]} \left\{ y_{T-2}U(c) + y_{T-1} (W_{T-2} - c)^\mu Q_{\xi_{T-2}}^\mu(\boldsymbol{\theta}_{\xi_{T-2}}^{(\mu)}) \right\} \\ &= \frac{y_{T-2}}{\mu} \max_{c \in [0, W_{T-2}]} \left\{ c^\mu + \frac{y_{T-1}}{y_{T-2}} Q_{\xi_{T-2}}^{\mu(1)} (W_{T-2} - c)^\mu \right\} \\ &= \frac{y_{T-2}}{\mu} \max_{c \in [0, W_{T-2}]} \left\{ c^\mu + L_{\xi_{T-2}}^{1-\mu} (W_{T-2} - c)^\mu \right\} \\ &= y_{T-2} \frac{W_{T-2}^\mu}{\mu} (1 + L_{\xi_{T-2}})^{1-\mu} \end{aligned}$$

where the maximum is achieved at $\boldsymbol{\theta} = \boldsymbol{\theta}_{\xi_{T-2}}^{(\mu)}$ and $c = W_{T-2} (1 + L_{\xi_{T-2}})^{-1}$ according to Lemmas 3.1 and 4.1. This shows that the theorem holds for $t = T-2$. Now we assume it is true for t . Then, for $t-1$ we have (again, here $c_{t-1}(\xi_{t-1})$ and $\boldsymbol{\theta}_{t-1}(\xi_{t-1})$ are denoted by c and $\boldsymbol{\theta}$, respectively)

$$\begin{aligned}
& V_{t-1}(W_{t-1}, \xi_{t-1}) \\
&= \max_{c \in [0, W_{t-1}], \boldsymbol{\theta} \in \Theta} \{y_{t-1}U(c) + \mathbf{E}[V_t(W_t, \xi_t) | \xi_{t-1}]\} \\
&= \max_{c \in [0, W_{t-1}], \boldsymbol{\theta} \in \Theta} \left\{ y_{t-1}U(c) + \mathbf{E} \left[y_t \frac{W_t^\mu}{\mu} (1 + L_{\xi_t})^{1-\mu} \middle| \xi_{t-1} \right] \right\} \\
&= \max_{c \in [0, W_{t-1}], \boldsymbol{\theta} \in \Theta} \left\{ y_{t-1}U(c) + \mathbf{E} \left[y_t (W_{t-1} - c)^\mu (1 + L_{\xi_t})^{1-\mu} \frac{1}{\mu} (r_{0, \xi_{t-1}} + \mathbf{R}'_{\xi_{t-1}} \boldsymbol{\theta})^\mu \middle| \xi_{t-1} \right] \right\} \\
&= \max_{c \in [0, W_{t-1}], \boldsymbol{\theta} \in \Theta} \left\{ y_{t-1}U(c) + y_t (W_{t-1} - c)^\mu \mathbf{E} \left[(1 + L_{\xi_t})^{1-\mu} \middle| \xi_{t-1} \right] Q_{\xi_{t-1}}^\mu(\boldsymbol{\theta}) \right\} \\
&= \max_{c \in [0, W_{t-1}]} \left\{ y_{t-1}U(c) + y_t (W_{t-1} - c)^\mu \mathbf{E} \left[(1 + L_{\xi_t})^{1-\mu} \middle| \xi_{t-1} \right] \max_{\boldsymbol{\theta} \in \Theta} Q_{\xi_{t-1}}^\mu(\boldsymbol{\theta}) \right\} \\
&= \max_{c \in [0, W_{t-1}]} \left\{ y_{t-1}U(c) + y_t (W_{t-1} - c)^\mu \mathbf{E} \left[(1 + L_{\xi_t})^{1-\mu} \middle| \xi_{t-1} \right] Q_{\xi_{t-1}}^\mu(\boldsymbol{\theta}_{\xi_{t-1}}^{(\mu)}) \right\} \\
&= \frac{y_{t-1}}{\mu} \max_{c \in [0, W_{t-1}]} \left\{ c^\mu + \frac{y_t}{y_{t-1}} Q_{\xi_{t-1}}^{\mu(1)} \mathbf{E} \left[(1 + L_{\xi_t})^{1-\mu} \middle| \xi_{t-1} \right] (W_{t-1} - c)^\mu \right\} \\
&= \frac{y_{t-1}}{\mu} \max_{c \in [0, W_{t-1}]} \left\{ c^\mu + L_{\xi_{t-1}}^{1-\mu} (W_{t-1} - c)^\mu \right\} \\
&= y_{t-1} \frac{W_{t-1}^\mu}{\mu} (1 + L_{\xi_{t-1}})^{1-\mu}
\end{aligned}$$

where the maximum is achieved at $\boldsymbol{\theta} = \boldsymbol{\theta}_{\xi_{t-1}}^{(\mu)}$ and $c = W_{t-1} (1 + L_{\xi_{t-1}})^{-1}$ by Lemmas 3.1 and 4.1. This indicates that the theorem is valid for $t - 1$ and hence the theorem is proved by the principle of induction. \square

We now make several remarks regarding Theorem 4.1. First, the optimal investment strategy for the investment-consumption problem (12) is identical to that for the investment only problem (3) as long as the preference for consumption coincides with the preference for terminal wealth; i.e. both power utilities have the same parameters $\gamma = \mu$. This implies that the optimal investment and consumption strategy can be separated and that the economic implications on the optimal investment strategy stated after Theorem 3.1 apply to this model as well. Second, in contrast to the optimal investment strategy, the optimal consumption strategy depends not only on the current economic state and the future asset returns, but also on the current wealth level and the remaining time horizon. Third, at each event ξ_t , the investor should optimally consume a fraction $(1 + L_{\xi_t})^{-1}$ of his wealth with the remaining wealth being invested among the $J + 1$ assets according to portfolio $\boldsymbol{\theta}_{\xi_t}^{(\mu)}$. And the optimal fraction changes with the economic states at that time and with times. Fourth, if the investor perceives a higher rate of mortality in the following

time period, then he will increase his current consumption.² This is consistent with our intuition since with the greater likelihood of dying, the investor is faced with the earlier exit time and consequently he should increase the current consumption in order to maximize his utility from consumption. Fifth, as can be seen from both (18) and (21) that if the investor is optimistic about the future economy, then the investor will reduce the current consumption. This again is aligned with our intuition since with the anticipated higher asset returns, the investor is willing to invest more to fully exploit the future higher investment gains. Sixth, it can also be verified that³ for a given initial wealth W_0 , the optimal expected utility $V_0(W_0, \xi_0)$ of the investor increases with the planned time horizon T , as to be expected.

5 An Extended Investment-Consumption Problem

In this section we extend our analysis by considering an investor who derives utility both from consumption (i.e., from “living well”) and from terminal wealth (i.e., from “becoming rich” either alive or dead). We also demonstrate how the optimal solutions established in the last two sections can be used to solve the extended optimization problem presented in this section. Since the investor is interested in maximizing the expected utility of both consumption and terminal wealth, the expected (aggregate) utility is given by

$$\mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \right] + \mathbb{E} [u(W_{T \wedge \tau})],$$

where u and U are defined in (5) and (13), respectively. The objective of the investor boils down to solving the following optimization problem:

$$\max_{\substack{\theta_t(\xi_t) \in \Theta, c_t(\xi_t) \in [0, W_t(\xi_t)]; \\ \xi_t \in \mathcal{F}_t, t=0, \dots, T-1}} \left\{ \mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \right] + \mathbb{E} [u(W_{T \wedge \tau})] \right\}, \quad (22)$$

²Since the probability y_t that the investor survives until present time t could not be changed, when the mortality rate q_{t+1} in the following time period increases, the probability y_{t+1} that the investor survives until time $t+1$ decreases, implying y_{t+1}/y_t decreases. Hence L_{ξ_t} decreases according to (18) and hence $c_t(\xi_t)$ increases due to (21).

³For a planned horizon with T time periods, $L_{\xi_{T-1}} = 0$ by (17). For a planned horizon with $T+1$ time periods, $L_{\xi_{T-1}} = \left\{ \frac{y_T}{y_{T-1}} Q_{\xi_{T-1}}^{\mu(1)} \right\}^{1/(1-\mu)}$ by (18) and (17), which is nonnegative and hence not smaller than the former. Note that L_{ξ_t} increases with $L_{\xi_{t+1}}$.

subject to the budget constraint (2).

The above formulation of the optimization requires an investor to balance between two conflicting objectives. If the investor were to “enjoy life” by consuming more at each intermediate time period, then the terminal wealth (i.e. either in the form of retirement income or bequest) would be substantially penalized. On the other hand if the investor wishes to receive more terminal wealth, then this can be achieved at the expense of spending less at each time period. The key is therefore to maintain an equilibrium tradeoff between consumption at each time and the wealth at terminal time.

Let $V(W_0)$ denote the maximum value to optimization problem (22) of an investor with initial endowment W_0 . To obtain its optimal investment-consumption strategies, we proceed as follows. At time $t = 0$, we first divide the investor’s initial endowment W_0 into two nonnegative components $W_0^{(1)}$ and $W_0^{(2)}$ such that $W_0^{(1)} + W_0^{(2)} = W_0$. Then, based upon the initial wealth $W_0^{(1)}$ and utility u , we solve the investment only problem (3) of Section 3 and use $V_1(W_0^{(1)})$ to denote its maximum value. Similarly, using $W_0^{(2)}$ as the initial wealth and with utility U , we solve the investment-consumption problem (12) of Section 4 and use $V_2(W_0^{(2)})$ to denote its maximum value. Finally we show that the superposition of the investor’s allocations for these two problems lead to the optimal policy for problem (22) provided both $W_0^{(1)}$ and $W_0^{(2)}$ are chosen such that their “marginal expected utilities” (i.e. $V_1'(W_0^{(1)})$ and $V_2'(W_0^{(2)})$) are identical.

From Theorem 3.1, the maximum $V_1(W_0^{(1)})$ is achieved at a pair $(\boldsymbol{\theta}^{(1)}, c^{(1)}) := (\boldsymbol{\theta}^{(\gamma)}, 0)$. By denoting $W^{(1)}$ as the wealth process corresponds to the optimal strategy $\boldsymbol{\theta}_t^{(1)}$ and $\boldsymbol{\pi}^{(1)}$ as the process of the optimal amount of the wealth invested in assets $1, \dots, J$ (i.e., $\boldsymbol{\pi}_t^{(1)} := W_t^{(1)} \boldsymbol{\theta}_t^{(1)}$ for all t), we obtain

$$\begin{aligned} W_{t+1}^{(1)}(\xi_{t+1}) &= W_t^{(1)}(\xi_t) \left[r_{0,\xi_t}(\xi_{t+1}) + \mathbf{R}_{\xi_t}(\xi_{t+1})' \boldsymbol{\theta}_t^{(1)}(\xi_t) \right] \\ &= W_t^{(1)}(\xi_t) r_{0,\xi_t}(\xi_{t+1}) + \mathbf{R}_{\xi_t}(\xi_{t+1})' \boldsymbol{\pi}_t^{(1)}(\xi_t), \end{aligned} \quad (23)$$

where $\xi_{t+1} \subset \xi_t \in \mathcal{F}_t$, $t = 0, 1, \dots, T - 1$. Similarly, it follows from Theorem 4.1 that the maximum $V_2(W_0^{(2)})$ is attained at a pair $(\boldsymbol{\theta}^{(2)}, c^{(2)})$, where $\boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(\mu)}$ and $c^{(2)}$ is given by (21) with W_0 being replaced by $W_0^{(2)}$. By denoting $W^{(2)}$ as the wealth process corresponding to the optimal strategy $\boldsymbol{\theta}_t^{(2)}$ and $\boldsymbol{\pi}^{(2)}$ as the process of the optimal amount of the wealth invested in assets $1, \dots, J$ (i.e., $\boldsymbol{\pi}_t^{(2)} :=$

$(W_t^{(2)} - c_t^{(2)}) \boldsymbol{\theta}_t^{(2)}$ for all t), we have

$$\begin{aligned} W_{t+1}^{(2)}(\xi_{t+1}) &= \left(W_t^{(2)}(\xi_t) - c_t^{(2)}(\xi_t) \right) \left[r_{0,\xi_t}(\xi_{t+1}) + \mathbf{R}_{\xi_t}(\xi_{t+1})' \boldsymbol{\theta}_t^{(2)}(\xi_t) \right] \\ &= \left(W_t^{(2)}(\xi_t) - c_t^{(2)}(\xi_t) \right) r_{0,\xi_t}(\xi_{t+1}) + \mathbf{R}_{\xi_t}(\xi_{t+1})' \boldsymbol{\pi}_t^{(2)}(\xi_t), \end{aligned} \quad (24)$$

where $\xi_{t+1} \subset \xi_t \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$.

Let us now define

$$\hat{W}_t := W_t^{(1)} + W_t^{(2)}, \quad \hat{c}_t := c_t^{(2)}, \quad \hat{\boldsymbol{\pi}}_t := \boldsymbol{\pi}_t^{(1)} + \boldsymbol{\pi}_t^{(2)}, \quad \text{and} \quad \hat{\boldsymbol{\theta}}_t := \frac{\hat{\boldsymbol{\pi}}_t}{\hat{W}_t - \hat{c}_t}, \quad (25)$$

then summing (23) and (24) yields

$$\begin{aligned} \hat{W}_{t+1}(\xi_{t+1}) &= \left(\hat{W}_t(\xi_t) - \hat{c}_t(\xi_t) \right) r_{0,\xi_t}(\xi_{t+1}) + \mathbf{R}_{\xi_t}(\xi_{t+1})' \hat{\boldsymbol{\pi}}_t(\xi_t) \\ &= \left(\hat{W}_t(\xi_t) - \hat{c}_t(\xi_t) \right) \left[r_{0,\xi_t}(\xi_{t+1}) + \mathbf{R}_{\xi_t}(\xi_{t+1})' \hat{\boldsymbol{\theta}}_t(\xi_t) \right], \end{aligned} \quad (26)$$

for $\xi_{t+1} \subset \xi_t \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$. Consequently, \hat{W} is the wealth process corresponding to the investment-consumption strategy $(\hat{\boldsymbol{\theta}}, \hat{c})$.

According to Theorems 3.1 and 4.1, we know that

$$\mathbb{E} [u(W_{T \wedge \tau})] \leq \mathbb{E} \left[u(W_{T \wedge \tau}^{(1)}) \right] = V_1 \left(W_0^{(1)} \right)$$

and

$$\mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \right] \leq \mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t^{(2)}) \right] = V_2 \left(W_0^{(2)} \right).$$

Adding them gives

$$\mathbb{E} \left[\sum_{t=0}^{T \wedge \tau - 1} U(c_t) \right] + \mathbb{E} [u(W_{T \wedge \tau})] \leq V_1 \left(W_0^{(1)} \right) + V_2 \left(W_0^{(2)} \right),$$

so that the optimal value to our extended investment-consumption problem (22) is bounded from above as follows:

$$V(W_0) \leq V_*(W_0) := \max_{\substack{W_0^{(1)} + W_0^{(2)} = W_0 \\ W_0^{(1)}, W_0^{(2)} \geq 0}} \left\{ V_1 \left(W_0^{(1)} \right) + V_2 \left(W_0^{(2)} \right) \right\}. \quad (27)$$

If we find $(W_0^{(1)}, W_0^{(2)})$ at which the maximum $V_*(W_0)$ is achieved, then the total expected utility corresponding to the pair $(\hat{\boldsymbol{\theta}}, \hat{c})$ will be exactly equal to $V_*(W_0)$.

Therefore, we have shown that $V(W_0) = V_*(W_0)$ and that the pair $(\hat{\theta}, \hat{c})$ is optimal to problem (22).

The optimal solution $(W_0^{(1)}, W_0^{(2)})$ to the maximization problem in (27) is determined by the system

$$V_1'(W_0^{(1)}) = V_2'(W_0^{(2)}), \quad W_0^{(1)} + W_0^{(2)} = W_0, \quad W_0^{(1)}, W_0^{(2)} \geq 0. \quad (28)$$

From Theorems 3.1 and 4.1, we have

$$\begin{aligned} V_1(W_0^{(1)}) &= v_0(W_0^{(1)}, \xi_0) = \frac{(W_0^{(1)})^\gamma}{\gamma} \sum_{n=0}^T p_{T-n} Q_{\xi_0}^{(T-n)}, \\ V_2(W_0^{(2)}) &= V_0(W_0^{(2)}, \xi_0) = y_0 \frac{(W_0^{(2)})^\mu}{\mu} (1 + L_{\xi_0})^{1-\mu} \end{aligned}$$

with $y_0 = 1$. Hence, the system (28) has a unique solution $(W_0^{(1)}, W_0^{(2)})$ and we establish the following result:

Theorem 5.1 *For a fixed initial wealth W_0 , let $(W_0^{(1)}, W_0^{(2)})$ be the unique solution to the system (28), and let $W^{(1)}$ and $W^{(2)}$ be, respectively, the optimal wealth processes of problem (3) with initial wealth $W_0^{(1)}$ and problem (12) with initial wealth $W_0^{(2)}$. Then, the optimal investment-consumption strategy of the extended investment-consumption problem (22) is given by*

$$\hat{\theta}_t(\xi_t) = \frac{W_t^{(1)}(\xi_t)}{W_t^{(1)}(\xi_t) + W_t^{(2)}(\xi_t) - \hat{c}_t(\xi_t)} \theta_{\xi_t}^{(\gamma)} + \frac{W_t^{(2)}(\xi_t) - \hat{c}_t(\xi_t)}{W_t^{(1)}(\xi_t) + W_t^{(2)}(\xi_t) - \hat{c}_t(\xi_t)} \theta_{\xi_t}^{(\mu)}, \quad (29)$$

$$\hat{c}_t(\xi_t) = W_t^{(2)} (1 + L_{\xi_t})^{-1} \quad (30)$$

for $\xi_t \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$.

It is of interest to note that the optimal investment strategy of the extended investment-consumption problem is a weighted average of the optimal investment strategies from both the investment only problem and the investment-consumption problem. In the special case where the preferences for both consumption and terminal wealth are identical (i.e., $\gamma = \mu$), then the three optimal investment strategies coincide. Note also that in the extended problem, the optimal consumption strategy does not depend on the utility function of terminal wealth. More specifically, the

optimal consumption strategy in the investment-consumption problem by assuming initial wealth $W_0^{(2)}$ is also the optimal consumption strategy to the extended investment-consumption problem.

6 Conclusion

By incorporating uncertainties due to economic environment, asset returns, and mortality in a multiperiod setup, this paper analyzed three investment-consumption problems for a risk averse investor with power utilities: (i) an investment only problem that involves utility from only terminal wealth, (ii) an investment-consumption problem that involves utility from only consumption, and (iii) an extended investment-consumption problem that involves utility from both consumption and terminal wealth. Using standard dynamic programming approach, analytic solutions to these problems were derived.

We also pointed out some interesting economic implications arising from our analytical results and many of which are consistent with our intuition. For example, the investment only problem shown that the optimal investment strategy is not influenced by the mortality risk. Whether we take into account the mortality risk, the same investment strategy is adopted. For the investment-consumption problem, we also demonstrated how the mortality and the asset returns affect the current consumption and the investment strategy. Typically, a higher future mortality risk leads to a greater current consumption (and hence a lower investment) while a greater future expected investment returns implies a greater current investment (and hence a lower consumption). For the extended investment-consumption problem that involves utility from both consumption and terminal wealth, we also demonstrated how the solutions from the earlier two problems can be used to obtain the solution for the extended case. More specifically, the key to solving the extended investment-consumption problem hinges on first obtaining the solution to the system (28). Once the optimal partition $(W_0^{(1)}, W_0^{(2)})$ of the initial wealth W_0 is derived, these partitions are in turn assumed to be the initial wealth levels for the investment only problem and the investment-consumption problem discussed in Sections 3 and 4. The optimal solutions from these problems are then used to determine the optimal investment-consumption strategies for the extended problem. As a

consequence, the optimal investment strategy for the extended problem is a convex combination of the optimal investment strategies from respectively the investment only and investment-consumption problems while the optimal consumption corresponds to the optimal consumption from the investment-consumption problem but with initial wealth replaced by $W_0^{(2)}$.

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