Table 7.2 Results for chance constrained formulation (1)

Probability {return 2' 0}	α	Stock	Bond	Gamble	Expected return
0.50	0	-	-	1000.00	152.00
0.80	0.842	585.19	-	414.81	149.66
0.90	1.282	863.18	-	136.82	148.55
0.95	1.645	515.28	427.39	57.33	110.62
0.99	2.326	260.87	707.91	31.21	85.83

The probability determines the penalty function α. At a probability of 0.80, the one-tailed normal z-function is 0.842, and thus the chance constrained is:

0:148 S þ 0:060 B þ 0:152 G - 0:253*SQRT(0:014697S^2 þ0:000936SB -0:004444SG þ0:000155B^2 -0:000454BG þ 0:160791G^2

The only difference in the constraint set for the different rows of Table 7.2 is that α is varied. The affect is seen is that investment is shifted from the high risk gamble to a bit safer stock. The stock return has low enough variance to assure the specified probabilities given. Had it been higher, the even safer bond would have entered into the solution at higher specified probability levels.

Minimize Variance

With this chance constrained form, Hal is risk averse. He wants to minimize risk subject to attaining a prescribed level of gain. The variance-covariance matrix measures risk in one form, and Hal wants to minimize this function.

Min 0:014697S^2 þ 0:000936SB - 0:004444SG þ 0:000155B^2 - 0:000454BG þ 0:160791G^2

This function can be constrained to reflect other restrictions on the decision. For instance, there typically is some budget of available capital to invest.

S þ B þ G ::; 1000   for a $1000 budget

Finally, Hal only wants to minimize variance given that he attains a prescribed expected return. Hal wants to explore four expected return levels: $50/$1000 invested, $100/$1000 invested, $150/$1000 invested, and $200/$1000 invested. Note that these four levels reflect expected returns of 5, 10, 15, and 20 %.

$$0.148\,S + 0.06\,B + 0.152\,G \geq r \quad \text{where } r = 50,\ 100,\ 150,\ \text{and}\ 200$$

Solution Procedure

The EXCEL input file will start off with the objective, MIN followed by the list of variables. Then we include the constraint set. The constraints can be stated as you want, but the partial derivatives of the variables need to consider each constraint stated in less-than-or-equal-to form. Therefore, the original model is transformed to:

$$
\begin{array}{ll}
\text{Min } .014697S^2 + .000936SB - .004444SG + .000155B^2 - .000454BG + .160791G^2 & \\
\text{st } S + B + G \leq 1000 & \text{budget constraint} \\
\quad 0.148\,S + 0.06\,B + 0.152\,G \geq 50 & \text{gain constraint} \\
\qquad S,\ B,\ G \geq 0 &
\end{array}
$$

The solution for each of the four gain levels are given in Table 7.3:

The first solution indicates that the lowest variance with an expected return of $50 per $1000 invested would be to invest $20.25 in S (stocks), $778.56 in B (the bond fund), $1.90 in G (the risky alternative), and keeping the 199.29 slack. The variance is $100.564. This will yield an average return of 5 % on the money invested. Increasing specified gain to $100 yields the designed expected return of $100 with a variance of $2807.182. Raising expected gain to 150 yields the prescribed $150 with a variance of $43,872. Clearly this is a high risk solution. But it also is near the maximum expected return (if all $1000 was placed on the riskiest alternative, G, the expected return would be maximized at $152 per $1000 invested). A model specifying a gain of $200 yields an infeasible solution, and thus by running multiple models, we can identify the maximum gain available (matching the linear programming model without chance constraints). It can easily be seen that lower variance is obtained by investing in bonds, then shifting to stocks, and finally to the high-risk gamble option.

Maximize Probability of Satisfying Chance Constraint

The third chance constrained form is implicitly attained by using the first form example above, stepping up α until the model becomes infeasible. When the probability of satisfying the chance constraint was set too high, a null solution

Table 7.3 Results for chance constrained formulation (2)

Specified Gain	Variance	Stock	Bond	Gamble
≥ 50	100.564	20.25	778.56	1.90
≥ 100	2807.182	413.28	547.25	39.47
≥ 150	43,872	500.00	-	500.00
≥ 152	160,791	-	-	1000.00

Table 7.4 Results for chance constrained formulation (3)

α	Stock	Bond	Gamble	Expected return
3	157.84	821.59	20.57	75.78
4	73.21	914.93	11.86	67.53
4.5	38.66	953.02	8.32	64.17
4.8	11.13	983.38	5.48	61.48
4.9 and up	-	-	-	0

was generated (don’t invest anything—keep all the \$1000). Table 7.4 shows solutions obtained, with the highest α yielding a solution being 4.8, associated with a probability very close to 1.0 (0.999999 according to EXCEL).