# QFI QF Model Solutions
# Spring 2021

## 1.

**Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

**Learning Outcomes:**

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov's theorem in changing measures.

**Sources:**

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (pages 194-196, 203, 205, 218, 219, 221-224, 234)

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 10, 14)

QFIQ-113-17 Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition, 2009, Ch. 2

**Commentary on Question:**

*The focus of this question is understanding the differences and implications of real-world versus risk-neutral probability measures by applying Ito's lemma, Girsanov's theorem, and the Radon-Nikodym ("R-N") derivative. Candidates struggled to show this understanding, especially for parts (c) and (d). Detailed commentaries are listed underneath each part.*

**1. Continued**

**Solution:**

(a) Determine the market price of risk for all $t \leq 1$.

**Commentary on Question**:
*There is a typo in the question, where the "S" is missing in the process of* $dS_t$ *when* $0 \leq t \leq 0.5$. *The correct process should be* $dS_t = 0.05S_t dt + 0.2S_t dW_t$. *But most candidates identified the typo.*

*Overall, most candidates were able to calculate the correct market price of risk. Credits were also given to the answers using the wrong process as stated in the question.*

The market price of risk is defined as

$$\lambda_t = \frac{\mu_t - r_t}{\sigma_t}$$

where $\mu_t$ is the stock price drift rate, $r_t$ is the risk-free rate, and $\sigma_t$ is the stock price volatility. By plugging in the numbers, we get

$$\lambda_t = \begin{cases} \dfrac{0.05 - 0.01}{0.2} = 0.2 & if\ 0 \leq t \leq 0.5 \\ \dfrac{-0.05 - 0.01}{0.3} = -0.2 & if\ 0.5 < t \leq 1 \end{cases}$$

(b) Calculate $E^{\mathbb{P}}\left[S_1 | S_{0.5}\right]$.

**Commentary on Question**:

For $0.5 < t \leq 1$, we apply Ito's lemma and get

$$dlnS_t = \left(\mu_t - \frac{\sigma_t^2}{2}\right)dt + \sigma_t dW_t = -0.095\, dt + 0.3dW_t$$

which gives

$$lnS_t - lnS_{0.5} = -0.095(t - 0.5) + 0.3(W_t - W_{0.5})$$

or

$$S_t = S_{0.5}e^{-0.095(t-0.5)+0.3(W_t - W_{0.5})}$$

Hence

$$E^P[S_t|S_{0.5}] = S_{0.5}E^P[e^{-0.095(t-0.5)+0.3(W_t-W_{0.5})}] = S_{0.5}e^{-0.095(t-0.5)+0.045(t-0.5)}$$

which leads to $E^P[S_1|S_{0.5}] = S_{0.5}e^{-0.025}$.

## 1. Continued

(c) Derive the Radon-Nikodym derivative of the risk-neutral measure $\mathbb{Q}$ with respect to the real-world measure $\mathbb{P}$.

**Commentary on Question**:
*Candidates performed poorly on this question. One source for this question (Chin et al) has many typos related to the Radon-Nikodym derivative. (page 222-223, 225, 239, 240). Whereas the other source - page 194-196, 203, 205, 218, 219 and 234 of (Chin el al) have the correct Radon-Nikodym derivatives. Credit was given to answers using the wrong R-N derivative as stated in the incorrect source.*

*For candidates that provided a general form of the R-N derivative, a common mistake made was to use $\lambda_t$ with the incorrect sign on the first integral component.*

*Most candidates were not able to derive the correct derivative when* $0.5 < t \leq 1$.

**Solution 1** – Based on the correct R-N derivative form from the source page 218.

The Radon-Nikodym derivative is calculated as

$$Z_s = e^{-\int_0^s \lambda_t dW_t - \frac{1}{2}\int_0^s \lambda_t^2 dt}$$

where $\lambda_t$ is the market price of risk as $\lambda_t = \frac{\mu_t - r_t}{\sigma_t}$, we get

$$\lambda_t = \begin{cases} 0.2, & 0 \leq t \leq 0.5 \\ -0.2, & 0.5 < t \leq 1 \end{cases}$$

Plugging in the numbers, we get

$$\int_0^s \lambda_t dW_t = \begin{cases} 0.2W_s & if\ 0 \leq s \leq 0.5 \\ -0.2(W_s - 2W_{0.5}) & if\ 0.5 < s \leq 1 \end{cases}$$

and

$$\int_0^s \lambda_t^2 dt = 0.04s$$

Hence

$$Z_s = \begin{cases} e^{-0.2W_s - 0.02s} & if\ 0 \leq s \leq 0.5 \\ e^{0.2(W_s - 2W_{0.5}) - 0.02s} & if\ 0.5 < s \leq 1 \end{cases}.$$

Or

The Radon-Nikodym derivative is calculated as (using notation in Neftci)

$$Z_s = e^{\int_0^s X_t dW_t - \frac{1}{2}\int_0^s X_t^2 dt}$$

where $X_t = -\lambda_t = \frac{r_t - \mu_t}{\sigma_t}$, we get

$$X_t = \begin{cases} -0.2, & 0 \leq t \leq 0.5 \\ 0.2, & 0.5 < t \leq 1 \end{cases}$$

**1. Continued**

Plugging in the numbers, we get

$$\int_0^s X_t dW_t = \begin{cases} -0.2W_s & if\ 0 \le s \le 0.5 \\ 0.2(W_s - 2W_{0.5}) & if\ 0.5 < s \le 1 \end{cases}$$

and

$$\int_0^s X_t^2 dW_t = 0.04s$$

Hence

$$Z_s = \begin{cases} e^{-0.2W_s - 0.02s} & if\ 0 \le s \le 0.5 \\ e^{0.2(W_s - 2W_{0.5}) - 0.02s} & if\ 0.5 < s \le 1 \end{cases}$$

**Solution that will not get credit for future sittings** – Based on the wrong R-N derivative from page 223 of Chin et al, where $dW_t$ and $dt$ were swapped. The errors will be published in the ERRATA list before the next sitting.

The Radon-Nikodym derivative is calculated as

$$Z_s = e^{-\int_0^s \lambda_t dt - \frac{1}{2}\int_0^s \lambda_t^2 dW_t}$$

where $\lambda_t$ is the market price of risk. Plugging in the numbers, we get

$$\int_0^s \lambda_t dt = \begin{cases} 0.2s & if\ 0 \le s \le 0.5 \\ 0.2(1-s) & if\ 0.5 < s \le 1 \end{cases}$$

and

$$\int_0^s \lambda_t^2 dW_t = 0.04W_s$$

Hence

$$Z_s = \begin{cases} e^{-0.2s - 0.02W_s} & if\ 0 \le s \le 0.5 \\ e^{-0.2(1-s) - 0.02W_s} & if\ 0.5 < s \le 1 \end{cases}$$

(d) Show that $\left\{S_t e^{-0.01t} : 0 \le t \le 1\right\}$ is a $\mathbb{Q}$-martingale.

**Commentary on Question**:
*Candidates had the most difficulty with this part. Many were able to prove the no drift condition, but failed to mention Girsanov's theorem or the R-N derivative as justification for the substitution of a different standard Wiener process under an equivalent measure. A complete response should demonstrate and justify the relationship between the two Wiener processes.*

For ease of presentation, we use $r_t, \mu_t$, and $\sigma_t$ to denote the risk-free rate, the drift rate of the stock price, and the volatility of the stock price. Let $Y_t = S_t e^{-\int_0^t r_u du}$.

# 1. Continued

Applying Ito's lemma on $Y_t$ and using the fact that

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

we obtain

$$\begin{aligned} dY_t &= e^{-\int_0^t r_u du} dS_t - r_t Y_t dt \\ &= e^{-\int_0^t r_u du}(\mu_t S_t dt + \sigma_t S_t dW_t) - r_t S_t e^{-\int_0^t r_u du} dt \\ &= e^{-\int_0^t r_u du}(\mu_t S_t dt - r_t S_t dt + \sigma_t S_t dW_t) \\ &= \sigma_t S_t e^{-\int_0^t r_u du}\left(dW_t + \frac{\mu_t - r_t}{\sigma_t} dt\right) \end{aligned}$$

Now let

$$\widetilde{W}_t = W_t + \int_0^t \frac{\mu_u - r_u}{\sigma_u} du$$

By Girsanov's theorem, there exists an equivalent measure defined by the R-N derivative, so that $\widetilde{W}_t$ is a standard Wiener process on the same filtration.

We then have

$$dY_t = \sigma_t S_t e^{-\int_0^t r_u du} d\widetilde{W}_u$$

Since $dY_t$ does not have the $dt$ term, $Y_t$ is a martingale under the risk-neutral measure $\mathbb{Q}$.

(e) Comment on whether each statement is true or not.

**Commentary on Question**:
*To receive full credit, candidates needed to explain the reasoning behind their assessment of the trueness of each statement. Most candidates simply repeated the statement, so only partial credit could be given.*

Statement A: This statement is true because two measures are equivalent if they have the same set of zero probabilities.

Statement B: This statement is false. The probability depends on the real probabilities and the real growth rate.

## 2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

**Learning Outcomes:**

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1f) Understand and apply Jensen's Inequality.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

**Sources:**

Problems and Solutions in Mathematical Finance, Chin
An introduction to Mathematics of Financial Derivatives, Nefci
QFIQ-113-17

**Commentary on Question:**

*Only a few candidates get full marks on this question. For part (b), some candidates failed to work out the integral. For part (c), only a few candidates correctly stated the triangle-inequality. Most of candidates were able to get part (d)*

**Solution:**

(a) List the criteria for the stochastic process $V_t$ to be a sub-martingale with respect to $(\Omega, \mathcal{F}, \mathbb{P})$.

**Commentary on Question**:
*Quite a few candidates misstated the 1st criteria with the incorrect filtration.*

The three criteria for $0 \le s \le t \le T$ are:
$E^{\mathbb{P}}(V_t|\mathcal{F}_s) \ge V_s$;

The inequality holds almost surely.
$E^{\mathbb{P}}[|V_t|] < \infty$;

And the 3rd criterion is that $V_t$ is $\mathcal{F}_t$-measurable

(b) Evaluate $Var^{\mathbb{P}}\left[|W_t|\right]$.

**Commentary on Question**:
*Most candidates failed to work out the integrals.*

$E^{\mathbb{P}}[|W_t|] = \int_{-\infty}^{\infty} |w| \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw$
$E^{\mathbb{P}}[|W_t|] = 2\int_{0}^{\infty} |w| \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw$

## 2. Continued

$$E^{\mathbb{P}}[|W_t|] = -2\int_0^\infty \frac{d}{dw}\left(\frac{\sqrt{t}}{\sqrt{2\pi}}e^{-\frac{w^2}{2t}}\right)dw = 2\frac{\sqrt{t}}{\sqrt{2\pi}} = \sqrt{\frac{2t}{\pi}}$$

$Var^{\mathbb{P}}[|W_t|] = E^{\mathbb{P}}[|W_t|^2] - E^{\mathbb{P}}[|W_t|]^2$ (definition of variance)
$Var^{\mathbb{P}}[|W_t|] = E^{\mathbb{P}}[W_t^2] - E^{\mathbb{P}}[|W_t|]^2$ $(as\ E^{\mathbb{P}}[|W_t|^2] = E^{\mathbb{P}}[W_t^2])$
$Var^{\mathbb{P}}[|W_t|] = t - 2t/\pi$

(c) Prove that $|W_t|$ is a non-negative sub-martingale.

**Commentary on Question**:
*Not many candidates are able show the convexity using the triangle-inequality.*

Using part (a), we will demonstrate each of the 3 criteria:

$|W_t|$ is clearly $\mathcal{F}_t$-measurable
From part (b), we have $E^{\mathbb{P}}\big[\big||W_t|\big|\big] = E^{\mathbb{P}}[|W_t|] = \sqrt{\frac{2t}{\pi}} < \infty$
In order to use conditional jensen's inequality, we need to establish that $\psi()$ = abs() is a convex function on $\mathbb{R}$. Let $\theta \in [0,1]$ $and\ x, y \in \mathbb{R}$

Using the triangle-inequality,

$|\theta x + (1-\theta)y| \le |\theta x| + |(1-\theta)y| = \theta|x| + (1-\theta)|y|$

Hence, we have shown abs() is convex

Applying the conditional Jensen inequality,

$E^{\mathbb{P}}(|W_t||\mathcal{F}_s) \ge \left|E^{\mathbb{P}}(W_t|\mathcal{F}_s)\right| = |W_s|$, since $W_t$ is a martingale

Hence $|W_t|$ is a sub-martingale. It is non-negative as $|W_t| \subseteq \mathbb{R}^+$

(d) Determine integer $k$ that makes $W_t^k$ a martingale.

**Commentary on Question**:
*Most of the candidates got this part right. Some did not provide the k = 0 solution.*

By Ito's Lemma, $dW_t^k = kW_t^{k-1}dW_t + \frac{1}{2}k(k-1)W_t^{k-2}(dW_t)^2$
$= kW_t^{k-1}dW_t + \frac{1}{2}k(k-1)W_t^{k-2}dt$
We need drift term to be zero to make the process a martingale.
When k=0 or 1, the drift term=0.

So if k=0, 1, then the process is a martingale.

## 3.

**Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

**Learning Outcomes:**

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

**Sources:**

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Chapters 6, 8, 9

**Commentary on Question:**

*This question tests candidates' knowledge of Ito's lemma, Ito's isometry, martingales, and basic properties of Brownian Motion. Most candidates did well on this question. Some candidates did not state what rules and formulas they were applying to from step to step.*

**Solution:**

(a) Derive $E\left[W_s^3 W_t\right]$ for $t > s$.

**Commentary on Question**:

*Most candidates did well on this part. The few candidates who did badly tried to decompose the wrong term and failed to state the independence of $W_s^3$ and $W_t - W_s$.*

By the properties of Brownian motion, we have

$E[W_s^3 W_t]$

$= E[W_s^3(W_s + W_t - W_s)]$

$= E[W_s^4] + E[W_s^3(W_t - W_s)]$

$= E[W_s^4] + E[W_s^3]E[(W_t - W_s)]$

$= E[W_s^4]$

Let $Z = \frac{W_s}{\sqrt{s}}$, which is a standard normal distribution. Then $E[Z^4] = 3$.

This gives

$E[W_s^4] = s^2 E[Z^4] = 3s^2$

(b) Determine the value of $c$ such that $W_t^3 - ctW_t$ is a martingale.

**Commentary on Question**:

*Most candidates did well on this part. Some candidates pursued the alternate solution of using Ito's lemma and setting the drift term to 0.*

# 3. Continued

Let $M_t = W_t^3 - ctW_t$. Then we have

$E[M_t|F_s] = E[(W_s + W_t - W_s)^3|F_s] - ctE[W_t|F_s]$

$= E[W_s^3|F_s] + 3E[W_s^2(W_t - W_s)|F_s] + 3E[W_s(W_t - W_s)^2|F_s] + E[(W_t - W_s)^3|F_s] - ctE[W_s|F_s] - ctE[W_t - W_s|F_s]$

$= W_s^3 + 0 + 3W_s(t - s) + 0 - ctW_s + 0$

$= M_s$ if c = 3

(c) Show that $X_t = \int_0^t W_u du$ is not a martingale.

**Commentary on Question**:
*Most candidates did well on this part. However, some candidates mistook the integral $\int_0^t W_u du$ for $\int_0^t W_u\, dW_u$ and said it is a martingale. Some candidates failed to apply stochastic integrals clearly and effectively to show that $X_t$ is not a martingale.*

By Product Rule, we have

$$X_t = tW_t - \int_0^t u\, dW_u$$

Let $s \leq t$. Since the Ito integral is a martingale, we have

$$E[X_t|F_s] = E[tW_t|F_s] - E\left[\int_0^t ud\, W_u|F_s\right] = tW_s - \int_0^s ud\, W_u = X_s + (t - s)W_s \neq X_s$$

for t>s. Hence $X_t$ is not a martingale.

(d) Calculate

(i) $E[V^2]$

(ii) $E[VY]$

**Commentary on Question**:
*Most candidates did well on this part. Some candidates lost points for not mentioning Ito's Isometry in part (i) or not stating the independence of V and G in part (ii).*

By Ito's isometry

(i)
$E[V^2] = \int_0^1 e^{-2s} ds = \frac{1}{2}(1 - e^{-2})$

## 3. **Continued**

(ii)
Let $G = \int_1^2 e^{-s} dW_s$
$Y = V + G$

$E[VY] = E[V(V + G)]$
Since V and G are independent,
$= E[V^2] + E[VG]$
$= E[V^2] + E[V]E[G]$
$= E[V^2]$
$= \frac{1}{2}(1 - e^{-2})$

**4.** **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

**Learning Outcomes:**

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1d) Understand and apply Ito's Lemma.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

**Sources:**

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Ch. 9, 10, 11, 15

**Commentary on Question:**

*The objective in this question was to test Ito's Lemma as applied to the valuation of derivatives on a security that is driven by a Weiner Process. Most candidates performed above average and partial credit was given for answers with calculation errors or missing steps.*

**Solution:**

(a) Show, using Ito's lemma, that $\sigma = 0.3$.

**Commentary on Question**:

*Candidates performed well on this question. An alternative solution was also accepted.*

Let $V = S^c$. Find the partial derivatives:

- $\frac{\partial V}{\partial S} = (S)^{c-1}c = VS^{-1}c$
- $\frac{\partial^2 V}{\partial S^2} = (S)^{c-2}c(c-1) = VS^{-2}c(c-1)$
- $\frac{\partial V}{\partial t} = 0$

## 4. **Continued**

Apply Ito's Lemma:

$$dV = \frac{\partial V}{\partial S}(dS) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial t}(dt)$$
$$= (VS^{-1}c)(0.045\,S\,dt + \sigma S dW_t) + \frac{1}{2}\left(VS^{-2}c(c-1)\right)(\sigma S\,dW_t)^2 + 0$$
$$= (Vc)(0.045\,dt + \sigma\,dW_t) + \frac{1}{2}\left(Vc(c-1)\right)(\sigma^2\,dt)$$
$$= V\left[0.045c + \frac{1}{2}c(c-1)\sigma^2\right]dt + Vc\sigma dW_t$$
$$\frac{dV}{V} = \left[0.045\,c + \frac{1}{2}c(c-1)\sigma^2\right]dt + c\sigma\,dW_t$$
$$\frac{dS^c}{S^c} = \left[0.045\,c + \frac{1}{2}c(c-1)\sigma^2\right]dt + c\sigma\,dW_t$$

Compare the coefficient of $dt$ and $dW_t$:

- $0.045\,c + \frac{1}{2}c(c-1)\sigma^2 = 0.18$
- $c\sigma = 0.6 \Rightarrow c = \frac{0.6}{\sigma}$

Substitute the second equation into the first:

$$0.045\left(\frac{0.6}{\sigma}\right) + \frac{1}{2}\left(\frac{0.6}{\sigma}\right)\left(\frac{0.6}{\sigma} - 1\right)\sigma^2 = 0.18$$

This can be written as

$$\left(\frac{0.027}{\sigma}\right) + 0.3\left(\frac{0.6}{\sigma} - 1\right)\sigma = 0.18$$

or $\sigma^2 = 0.09$ which implies $\sigma = 0.3$ since it is positive.

*Alternative Solution:*

Using the solution formula to the Geometric Brownian Motion:

$$(S_t)^c = (S_0)^c e^{c\left(r-\frac{1}{2}\sigma^2\right)t + c\sigma W_t}$$
$$= (S_0)^c e^{c\left(.045-\frac{1}{2}\sigma^2\right)t + c\sigma W_t}$$

$$(S_t)^c = (S_0)^c e^{\left(0.18-\frac{1}{2}0.6^2\right)t + 0.6W_t}$$
$$= (S_0)^c e^{0.6W_t}$$

Compare the coefficient of $dt$ and $dW_t$:

- $0.045\,c - \frac{1}{2}c\sigma^2 = 0$
- $c\sigma = 0.6$

# 4. **Continued**

Solve the system of equations:

$\sigma^2 = 2 * 0.045$

$\sigma^2 = 0.09$ which implies $\sigma = 0.3$ since it is positive

$$c = \frac{0.6}{\sigma} = \frac{0.6}{0.3} = 2$$

(b) Calculate the time-0 no-arbitrage price of this derivative security.

**Commentary on Question**:

*Candidates performed ok on this part of the question. Common mistakes were to forget the discount term when computing the time-0 no-arbitrage price and failing to convert to a standard normal random variable before applying the formula given in the question.*

Use the following equivalency:

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t \Leftrightarrow S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t}$$

$$\frac{dS_t}{S_t} = 0.045\,dt + 0.3\,dW_t \Leftrightarrow S_t = 1e^{\left(0.045-\frac{1}{2}(0.3)^2\right)t+0.3W_t} = e^{0.3W_t}$$

Thus, we have $S_3 = e^{0.3W_3}$, where $W_3 \sim N(0,3)$.

The expected value of the derivative security under the risk-neutral probability measure is:

$$\begin{aligned} E[S_3(\ln S_3)^2] &= E[e^{0.3W_3}(\ln e^{0.3W_3})^2] \\ &= E[e^{0.3W_3}(0.3W_3)^2] \\ &= 0.09E[e^{0.3W_3}W_3^2] \end{aligned}$$

Since $Z \sim N(0,1)$, it follows that $W_3 = Z\sqrt{3}$, and thus:

$$\begin{aligned} E[S_3(\ln S_3)^2] &= 0.09E\left[e^{0.3Z\sqrt{3}}\left(Z\sqrt{3}\right)^2\right] \\ &= 0.09(3)E\left[e^{0.3\sqrt{3}\cdot Z}Z^2\right] \\ &= 0.09(3)\cdot\left(1+\left(0.3\sqrt{3}\right)^2\right)e^{0.5\left(0.3\sqrt{3}\right)^2} \\ &= 0.39246 \end{aligned}$$

The time-0 no-arbitrage price is:

$$E[S_3(\ln S_3)^2]\cdot e^{-3r} = 0.39246\cdot e^{-3(0.045)} = 0.3429$$

## 5. Learning Objectives:
2. The candidate will understand the fundamentals of fixed income markets and traded securities.

**Learning Outcomes:**

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

**Sources:**

Fixed Income securities (Chapter5,6), Veronesi

**Commentary on Question:**

*The comparison of interest rate futures and forwards went very well, and the computation of their prices and values in part b) was relatively good also. The purchase price was missing in part c) and it was taken into consideration in the grading (see comment in part c)).*
*The Put/Call Parity concept of part d) was well understood but there were some minor errors it the formula itself or in the computation of some of its components.*

*The candidates did not perform well at all on part e). Maybe they have not understood well the question or have not analyzed or studied such a more elaborate strategy which was a new subject.*

**Solution:**

(a) Compare interest rate futures and forwards, and discuss the advantages/disadvantages of futures compared to forwards.

**Commentary on Question**:

*The only item missing in general in answering the question was the definition of both contracts in terms of selling/buying the prescribed security, and how they can be settled.*

Futures: A futures contract is similar to a forward contract, in which the counterparty short the contract agrees to sell a prespecified security on a prespecified date and at a prespecified price to the counterparty long the contract. The latter agrees to buy the security and to pay the prespecified price. Some futures contracts are cash settled, meaning that no exchange of security actually takes place.
Characteristics of futures contracts are that they are:
• Traded in regulated exchanges.
• Standardized: The maturity of the contracts as well as the delivery securities are decided by the exchange.
• Marked-to-market: Profits and losses accrue to the counterparties on a daily basis.

**5.** **Contined**

Disadvantages:
Basis risk. The available maturity of the bond, or the particular instrument may not be the exact instrument to hedge all of the risk. Using a forward rate agreement, a firm could perfectly hedge the risk. Using futures, the firm would retain some residual risk, as the available instruments (the Eurodollar futures, based on the 3-month LIBOR) is not perfectly correlated with the interest rate to hedge.
Tailing of the Hedge. The cash flows arising from the futures position accrue over time, which implies the need of the firm to take into account the time value of money between the time at which the cash flow is realized and the maturity of the hedge position (maturity T in the example).

Advantages:
Liquidity. Because of their standardization, futures are more liquid than forward contracts, meaning that it is easy to get in and out of the position.
Credit Risk. The existence of a clearinghouse guarantees performance on futures contracts, while the same may not be true for forward contracts. The clearing house hedges itself through the mark-to-market provision.

(b) Compute the value of the forward contract at time 0 and year 1.

$$Z(t,T) = e^{-r(t,T)(T-t)}$$

Z(0,2) = exp(-2.48%*2) = 0.951610
Z(0,3) = exp(-2.46%*3) = 0.928857
Z(0,4) = exp(-2.51%*4) = 0.904476
Z(0,5) = exp(-2.51%*5) = 0.882056
Z(0,6) = exp(-2.59%*6) = 0.856073
Z(0,7) = exp(-2.59%*7) = 0.834185

Z(1,2) = exp(-0.16%*1) = 0.998401
Z(1,3) = exp(-0.16%*2) = 0.996805
Z(1,4) = exp(-0.18%*3) = 0.994615
Z(1,5) = exp(-0.29%*4) = 0.988467
Z(1,6) = exp(-0.29%*5) = 0.985605
Z(1,7) = exp(-0.49%*6) = 0.971028

$$P_c^{fwd}(0,T,T^*) = \frac{c}{2} * \sum_{i=1}^{n} P_Z^{fwd}(0,T,T_i) + P_Z^{fwd}(0,T,T_n) =$$

$$\frac{c*100}{2} \sum_{i=1}^{n} F(0,T,T_i) + 100 * F(0,T,T_n)$$

# 5. Contined

[** Note that the formula is for a semi-annual coupon bond, however the question here is for an annual coupon bond so before applying the formula,  we need to remove 2 from the formula. ]

where $$F(t,T_1,T_2)=\frac{Z(t,T_2)}{Z(t,T_1)}$$

Forward price at time 0 =
4%*(100*(0.928857+0.904476+0.882056+0.856073+0.834185)+100*0.834185)/0.951610
= 106.18

Forward price at time 1 =
4%*(100*(0.996805+0.994615+0.988467+0.985605+0.971028)+100*0.971028)/0.998401
= 117.04

Value of the forward contract at initiation is 0.
*Value of the forward contract at year 1 is* (117.04 – 106.18)*0.998401= 10.84

(c) Yesterday you bought a 5-year Treasury note future expiring in 2 years and today the future price drops to $100. The future contract size is $1,000,000. Your broker requires initial Margin: $1,485 (per contract); Maintenance Margin: $1,110 (per contract).

Calculate the cash flow today.

$$P\,\&\,L=k*contract*\left[P^{fut}(t,T)-P^{fut}(t-dt,T)\right]$$

[The purchase price is missing in the stem. So candidates who tried to answer the question and gave a reasonable answer received full credit.]

(d) You are given the following data at time 0:

- The 6-month zero coupon bond is priced at $98.24
- The 9-month zero coupon bond is priced at $97.21
- Call option (European) on the 3-month Treasury bill with maturity in 6 months and strike price of $99.12 is priced at $0.2934
- Put option (European) on the 3-month Treasury bill with maturity in 6 months and strike price of $99.12 is priced at $0.2044

**5. Contined**

Explain why the above securities are priced incorrectly.

$$Call(K) = Put(K) + Z(0,T_1) \times \left(P^{fwd}(0,T_1,T_2) - K\right)$$

0.2934 > 0.2044+0.9824*(97.21/98.24*100-99.12) = 0.038912
No, the security is not priced correctly.

(e) Describe a strategy to take advantage of the arbitrage opportunity:

(i) if the 3-month Treasury bill price is higher than the strike price.

(ii) if the 3-month Treasury bill price is lower than the strike price.

**Commentary on Question**:
*This is a new subject that may be was not enough studied, understood, or practiced.*

Buy the put option on the 3 moth treasury bill with maturity in 6-months and strike price of $99.12 at $0.2044
Buy 9-month zero coupon bond with notional of $100 at $97.21
Sell 6-month zero coupon bond with notional of $99.12 at $97.38 (99.12*0.9824)
Sell the call option on the 3 moth treasury bill with maturity in 6-months and strike price of $99.12 at $0.2934

At time 0
The net cash flow is -0.2044-97.21+97.38+0.2934 = $0.1323

Scenario 1: If the 3 month treasury bill price is higher than $99.12 at month 6.
At month 6
The call option is exercised to receive 99.12 for selling the 3 month treasury bill
Put option expires worthless.
Pay 99.12 for the 6-month zero coupon bond
No cash flow from the 9-month zero coupon bond.
Net cash flow at month 6 is 0.

At month 9
Receive $100 from the 9-month zero coupon bond maturity.
Pay $100 from the 3-month zero coupon bond sold at month 6.
Net cash flow is 0.

Scenario 2: If 3 month treasury bill price is lower than $99.12 at month 6
At month 6
Exercise the put option to sell the 3 month treasury bill, receiving 99.12
The call option expires out of the money

## 5. **Contined**

Pay 99.12 for the 6-month zero coupon bond
No cash flow from the 9-month zero coupon bond.

At month 9
Receive $100 from the 9-month zero coupon bond maturity.
Pay $100 from the 3-month zero coupon bond sold at month 6.
Net cash flow is 0.

## 6. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

**Learning Outcomes:**

(2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

**Sources:**

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch 4

QFIQ-121-20: A Guide to Duration, DV01, and Yield Curve Risk Transformations, pp. 1-28

**Commentary on Question:**

*Commentary listed underneath question component.*

**Solution:**

(a) Calculate the value of *k* based on Analyst A's proposal.

**Commentary on Question**:
*This part of the question was trying to test a candidate's comprehension of factor duration and ability to use it. Most candidates did well. Some candidates did not get the concept of dollar duration. Partial credits were given for work shown on intermediate calculations.*

*Total portfolio value* $V = P + k \times P_z$
*In order to implement duration hedging, we need* $dV = 0$
$D_p \times P + k \times D_z \times P_z(0,T) = 0$
*First calculate duration of the 5-year semi-annual coupon bond.*
$D_p = \sum_{i=1}^{10} w_i \times T_i = 4.4304$
*Duration of a 5-year zero coupon bond ($D_z$) is 5*
$P = \sum_{i=1}^{10} Z(0,T_i)_i \times CF_i = 111.23$ *million*
$P_z = 100 \times 0.8395 = 83.95$ *million*
*Substitute into the original formula,* $k = -D_p \times P / (D_z \times P_z(0,T))$
$= -(4.4304 \times 111.23) / (5 \times 83.95) = -1.1741$

(b) Calculate the DV01 of the portfolio consisting of the original bonds plus hedging strategy calculated in part (a).

**6. Continued**

**Commentary on Question**:
*Some candidates were not familiar with the formula of DV01. Partial credits were given if the candidates correctly calculate the price of the portfolio at the current yield plus or minus 0.1%.*

*DV01* = $\frac{P\ (current\ yield + 0.1\%) - P\ (current\ yield - 0.1\%)}{0.2}$
*(i.e. substituting k for correct portfolio value)*
*Pick the correct values from the table showing Z(0,5) at different yields, i.e. 3.60% and 3.40%*
*P(at 3.60%) = 110.74 - 1.1741 x 83.53*
*P(at 3.40%) = 111.72 - 1.1741 x 84.37*
*DV01 portfolio =*
*(110.74 - 1.1741 x 83.53 - 111.72 + 1.1741 x 84.37) / 0.2 = $0.031*

(c) Construct a hedging portfolio based on Analyst B's proposal.

**Commentary on Question**:
*This part of the question intends to test a candidate's ability to apply the concept of a duration and convexity hedged portfolio. Some candidates did not factor in price of the instruments when solving for the hedging units. Partial credits were given for calculating the components correctly.*

*Total value of the porposed portfolio is* $V = P + k_1 \times P_1 + k_2 \times P_2$, *where* $P_1$ *and* $P_2$ *represent prices of the 2-year zero coupon bond and 5-year zero coupon bond each with 100 million of par value.*
*Then we need,*
$k_1 \times D_1 \times P_1 + k_2 \times D_2 \times P_2 = -D \times P$ *(Delta Hedging)*
$k_1 \times C_1 \times P_1 + k_2 \times C_2 \times P_2 = -C \times P$ *(Convexity Hedging)*
$D_1 = 2$; $C_1 = 2^2 = 4$
$P_1 = 100 \times 0.9324 = 93.24$ *million*
*From part a) (i)* $D_2 = 5$; $P_2 = 83.95$ *million*; $D = 4.4304$; $P = 111.23$ *million*
$C_2 = 5^2 = 25$
*Convexity of the coupon bond:* $C = \sum_{i=1}^{10} w_i \times T_i^2 = 21.1320$
*The solution of this system of two equations in two unknowns is,*
$k_1 = -\frac{P}{P_1} \times \left(\frac{D \times C_2 - C \times D_2}{D_1 \times C_2 - C_1 \times D_2}\right)$
$k_2 = -\frac{P}{P_2} \times \left(\frac{D \times C_1 - C \times D_1}{D_2 \times C_1 - C_2 \times D_1}\right)$
*Substitute the values into the above formulae and solve for* $k_1$ *and* $k_2$
$k_1 = -0.2028$
$k_2 = -1.0839$

(d) Identify two other hedging instruments (in addition to zero-coupon bonds) that your company can use to mitigate the risk of an upward shift in interest rates.

**6. Continued**

**Commentary on Question**:
*Most candidates were able to list the hedging instruments; however, full credits are given only when candidate lists the instrument along with identifying whether a long or short position can mitigate interest rate risk*

Examples of acceptable answers include:
*(i) short position in a forward rate agreement*
*(ii) long position of fixed-for-floating interest rate swap contract (pay fixed, receive floating)*
*(iii) long position of a cap on spot rate (interest rate option)*

(e) Describe the procedure to construct a factor neutral hedge.

**Commentary on Question**:
*Most candidates struggled to provide a complete solution to this question. Partial credit was given for identifying pieces of the solution.*

*Consider now a portfolio P with factor durations D1 and D2 with respect to level and slope factors $\varphi 1$ and $\varphi 2$, respectively*
*To implement factor neutrality, we need to select two*
*other securities, one for each factor we want to neutralize, in appropriate proportions.*
*For instance, we could use short- and a long-dated zero coupon bonds, denoted by PSz and PLz .*
*For each of these two bonds we can compute the factor durations.*
*In order to immunize the portfolio against changes in level and slope, we must choose an amount of short-term and long-term zero coupon bonds, kS and kL , such that the variation of the portfolio plus the two bonds is approximately zero.*

(f) Calculate factor duration of bond S and bond L by level, slope, and curvature.

**Commentary on Question**:
*Most candidates answered this question correctly. Some did not multiply the bonds by the appropriate time and were awarded partial credit if they got the beta only correct.*

*Recall that for zero coupon bonds, factor duration of zero coupon bound i with respect to factor j:*

$$D_{j,z} = (T_i - t) \times \beta_{ij}$$

*For bond S,*

$D_1 = 2x\ 1.0344 = 2.0688$

$D_2 = 2\ x\ -0.3507 = -0.7014$

$D_3 = 2\ x\ 0.3228 = 0.6456$

*For bond L,*

$D_1 = 7\ x\ 1.0111 = 7.0777$

$D_2 = 7\ x\ 0.5208 = 3.6435$

$D_3 = 7\ x\ -0.1058 = -0.7406$

## 7. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(3c) Calibrate a model to observed prices of traded securities.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

**Sources:**

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Piertro, 2010, Chapter 14~15

**Commentary on Question:**

*The purpose of this question is to test candidates' understanding of the Vasicek model and calibration used in practice. Most of candidates understood the Vasicek model well but almost all candidates did not perform well in the calibration problem.*

**Solution:**

(a)

(i) Solve the stochastic differential equation.

(ii) Identify the distribution of $r_t$ by providing its mean and variance.

Consider $F(t, r_t) = e^{at} r_t$.

Since $\frac{\partial F}{\partial t} = a e^{at} r_t, \frac{\partial F}{\partial r} = e^{at}, \frac{\partial^2 F}{\partial r^2} = 0$, Ito's lemma gives us

$$dF = a e_t^{at} r_t dt + e^{at} dr_t = [a e^{at} r_t + e^{at}(\nu - a r_t)]dt + e^{at}\sigma dX_t$$
$$= e^{at}\nu dt + e^{at}\sigma dX_t$$
$$F(t, r_t) = F(0, r_0) + \nu \int_0^t e^{as} ds + \int_0^t e^{as}\sigma dX_s$$

where $F(0, r_0) = r_0$

$$\therefore r_t = e^{-at} r_0 + \nu \int_0^t e^{a(s-t)} ds + \sigma e^{-at}\int_0^t e^{as} dX_s = \mu_t + \sigma e^{-at}\int_0^t e^{as} dX_s$$

where $\mu_t = e^{-at} r_0 + \nu \int_0^t e^{a(s-t)} ds$

The mean of $r_t$ is $\mu_t$.

The variance is:

$\sigma^2 e^{-2at}\int_0^t e^{2as} dt$ (by Ito Isometry) $= \frac{\sigma^2 e^{-2at}}{2a}(e^{2at} - 1) = \frac{\sigma^2(1 - e^{-2at})}{2a}$

Also, it shows Gaussian distribution.

# 7. Continued

(b) Show that the limiting distribution of $r_t$ as $t$ approaches infinity is $N\left(\frac{\nu}{a},\frac{\sigma^2}{2a}\right)$

$$\lim_{t\to\infty} E[r_t] = \lim_{t\to\infty}\left[e^{-at}r_0 + \nu\int_0^t e^{a(s-t)}ds\right] = \lim_{t\to\infty}\left[e^{-at}r_0 + \frac{(1-e^{-at})\nu}{a}\right] = \frac{\nu}{a}$$
$$\lim_{t\to\infty} Var[r_t] = \lim_{t\to\infty}\frac{\sigma^2(1-e^{-2at})}{2a} = \frac{\sigma^2}{2a}$$

(c) Demonstrate that the interest rate, $r_{t+m}$, follows the same distribution. Hint: Use time frame $(m, t+m)$ from solution of part (a).

**Commentary on Question**:
*Quite a few candidates expressed $r_{t+\mathrm{m}}$ with an initial value of $r_0$ instead of $r_m$.. Then, they took a limit value as in part (b) to get the desired answer.*

Assume $r_m$ is stochastic, independent of the Brownian motion $X_t$. If we have that $r_m \sim N\left(\frac{\nu}{a},\frac{\sigma^2}{2a}\right)$, independent of $X_t$, then we have:

$$F(t+m, r_{t+m}) = F(m, r_m) + \nu\int_m^{t+m} e^{as}ds + \int_m^{t+m} e^{as}\sigma dX_s$$
$$\text{where } F(m, r_m) = r_m e^{am}$$
$$\therefore r_{t+m} = e^{-a(t)}r_m + \nu e^{-a(t+m)}\int_m^{t+m} e^{as}ds + \sigma e^{-a(t+m)}\int_m^{t+m} e^{as}dX_s$$
$$E[r_{t+m}] = e^{-a(t)}E[r_m] + \frac{\left(1-e^{-a(t)}\right)\nu}{a} = \frac{\nu}{a}$$
$$Var[r_{t+m}] = e^{-2a(t)}Var[r_m] + \frac{\sigma^2\left(1-e^{-2a(t)}\right)}{2a} = \frac{\sigma^2}{2a}$$

(d)

(i) Estimate the parameters for interest rate process above.

(ii) Describe for the estimation of arbitrage free parameters using the table below observed in the market.

From
$$dr_t = [\nu - ar_t]dt + \sigma dX_t$$
It can be written in discrete manner
$$r_{t+\delta} - r_t = -ar_t\delta + \nu\delta + \sigma\varepsilon_t\sqrt{\delta}, \varepsilon_t \sim N(0,1)$$
$$r_{t+\delta} = (1-a\delta)r_t + \nu\delta + \sigma\varepsilon_t\sqrt{\delta}, \varepsilon_t \sim N(0,1)$$

# 7. Continued

According to coefficient of regression from the hint,

$$\beta = 1 - a\delta,$$
$$\alpha = \nu\delta,$$
$$Var(r_{t+\delta}) = \sigma^2\delta$$

with $\delta = 0.25$ from the table

$$\beta = \frac{20 \cdot \sum_{i=1}^{20} r_{i-1} \cdot r_i - \sum_{i=1}^{20} r_i \cdot \sum_{i=1}^{20} r_{i-1}}{20 \cdot \sum_{i=1}^{20} r_{i-1}^2 - \left(\sum_{i=1}^{20} r_{i-1}\right)^2} = 0.089788$$

$$\alpha = \frac{\left(\sum_{i=1}^{20} r_i - \beta \sum_{i=1}^{20} r_{i-1}\right)}{20} = 0.037312$$

Therefore,

$$a = \frac{1-\beta}{\delta} = \frac{1 - 0.089788}{0.25} = 3.6408$$
$$\nu = \frac{\alpha}{\delta} = \frac{0.03737}{0.25} = 0.14927,$$
$$Var(r_{t+\delta}) = \frac{1}{20} \cdot \sum_{i=1}^{20} r_i^2 - \left(\frac{1}{20}\sum_{i=1}^{20} r_i\right)^2 = 0.000266$$
$$\sigma = \sqrt{\frac{Var(r_{t+\delta})}{\delta}} = \sqrt{\frac{0.000266}{0.25}} = 0.032628$$

For the arbitrgae free parameter estimation, it can be found by minimizing the errors between the arbitrage zero coupon bond prices in parametric formula and observed zero coupon bond prices.
For instance, $a^*, \nu^*$ can be searched by minimizing

$$J(a^*, \nu^*) = \sum_{i=1}^{n} \left(Z^{Vasicek}(0, T_i, a^*, \nu^*) - Z^{Data}(0, T_i)\right)$$

Each term in the parenthesis is the model's pricing error for each maturity $T_i$, that is, the distance between the model price and the data. If the model works well, each pricing error should be small, and thus also the sum of the pricing errors squared for nonlinear least square search.

**8.** **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3l) Demonstrate an understanding of the issues and approaches to building models that admit negative interest rates.

**Sources:**

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 p. 710, 716~717

Negative Interest Rates and Their Technical Consequences, AAE, 12/2016 p. 8, 9

QFIQ-116-17 Low Yield Curves and Absolute/Normal Volatilities p. 4~5, p. 7, p. 8, p. 9

**Commentary on Question:**

*This question was testing the candidates understanding of martingales, derivative pricing, and real-world application of different methodologies for calculating the derivative price. Most candidates did not do well with understanding martingales but improves with the pricing and real-world applications*

*Most candidates did not do well in the beginning but much better in later parts.*

**Solution:**

(a)

(i) Show that the forward price can also be computed as $F_A(t,T) = E_f^*\left(A_T \middle| \mathcal{F}_t\right)$ under the $T$-forward risk-neutral measure $\mathbb{Q}^T$.

(ii) Prove, using part (a)(i), that $\left\{F_A(t,T), t \geq 0\right\}$ is a martingale under the $T$-forward measure $\mathbb{Q}^T$.

**Commentary on Question**:

*The question was trying to test the concepts of alternative risk-neutral measure and martingale. Only a handful of candidates received full credits by demonstrating full understanding of the concepts by writing complete formulas.*

## 8. Continued

*Many candidates were able to list the requirements for a martingale but did not prove how the forward measure met each of the requirements. Only a few candidates used an alternative approach in using the Tower rule and most received full credit,*

From under the measure P to T- forward risk neutral measure:

$$B_t^{-1}A_t = E^P(B_T^{-1}A_T), A_t = B_t E^P(B_T^{-1}A_T)$$

$$A_t = e^{rt}E^P(e^{-rT}A_T) \to e^{-r(T-t)}E_f^*(A_T) = Z(t,T)E_f^*(A_T).$$

Using information set $\mathcal{F}_t$ at t,

$$F_A(t,T) = \frac{A_t}{Z(t,T)} = \frac{B_t E^P(B_T^{-1}A_T|\mathcal{F}_t)}{Z(t,T)} = \frac{Z(t,T)E_f^*(A_T|\mathcal{F}_t)}{Z(t,T)} = E_f^*(A_T|\mathcal{F}_t)$$

Hence $F_A(t,T) = E_f^*(A_T|\mathcal{F}_t)$ under the T-forward risk neutral measure.
Using the tower rule,

$$E_f^*(F_A(s,T)|\mathcal{F}_t) = E_f^*\left(E_f^*(A_T|\mathcal{F}_s)|\mathcal{F}_t\right) = E_f^*(A_T|\mathcal{F}_t) = F_A(t,T), t \le s \le T$$

Therefore, $\{F_A(t,T), t \ge 0\}$ is a martingale under T-forward measure $\mathbb{Q}^{\boldsymbol{T}}$.

(b) Calculate the value of this put option.

Hint: For a log-normally distributed variable $x$,

$$E\left[x^a\right] = E\left[x\right]^a e^{\frac{(a-1)a}{2}\sigma^2_{\log(x)}}, Var\left[\log x^a\right] = a^2 \cdot \sigma^2_{\log(x)}$$

*This was a simple calculation of a power put option. Many candidates received most credits if they determined the type of put option and converted it to the lognormal values. Some candidates did not recognize the correct option and had the calculations incorrectly. Very few candidates received full credit because of either the use of T in the calculation of d1 and d2 or incorrectly calculating the variance.*

$$E_f^*\left(r(0.5,0.75)^{\frac{1}{3}}\right) = f(0,0.5,0.75)^{\frac{1}{3}} e^{\frac{\left(\frac{1}{3}-1\right)*\frac{1}{3}}{2}\sigma_f^{fwd}(0.75)^2\cdot 0.5}$$

$$= 0.030250^{1/3} \times e^{-\frac{1}{9}\times 0.2^2\times 0.5} = 0.31089234$$

$$\sigma_T = \sqrt{V\left[r(0.5,0.75)^{\frac{1}{3}}\right]} = \sqrt{\left(\frac{1}{3}\right)^2 \sigma_f^{fwd}(0.75)^2 \cdot 0.5} = \sqrt{0.00222222}$$

$$= 0.04714045$$

$$d_1 = \frac{1}{\sigma_T}\log\left(\frac{E_f^*\left(r(0.5,0.75)^{\frac{1}{3}}\right)}{K}\right) + \frac{1}{2}\sigma_T = 0.035111, N(-d_1) = 0.4859$$

$$d_2 = d_1 - \sigma_T = -0.01203, N(-d_2) = 0.5048$$

**8. Continued**

Therefore, the value of put is given by:

$$100\ million\ \times 0.9845 \times \left(0.03^{\frac{1}{3}} \times 0.5048 - 0.31089234 \times 0.4859\right) =$$
$$0.5671\ million$$

(c) State two arguments to explain why low or negative interest rates were observed in 2008-2016 period in major markets.

**Commentary on Question**:
*This question was directly testing the facts from the materials. Many candidates received full credit for identifying the real-world occurrences that lead to lower interested rates post-2008. Some candidates struggled to identify the increase in savings*

1. A major contributor has been. ‘unconventional’ monetary policies introduced by many governments in response to the 2008 Global Financial Crisis. The line of reasoning highlights the desire of many governments to boost demand by lowering interest rates and thereby combat a weak economic environment.

2. Longer-term (multi-decade) trends that had been pushing down interest rates even prior to the Crisis and have continued to do so since then. This reasoning argues that shifting demographic profiles and other factors have increased the supply of saving and reduced the demand for borrowings. It argues that interest rates have fallen to address the supply/demand dynamics created by these factors.

(d) Consider volatilities in a low interest rate environment and provide answers for following.

(i) Describe the problem of using Black implied volatilities in interest rate sensitive derivative pricing and explain the advantages of using absolute/normal volatilities by describing practitioners’ view.

(ii) Describe the correlations between yield curve levels and volatilities for Black volatilities and absolute/normal volatilities.

(iii) Recommend the choice of volatilities in the current low interest rate environment. Justify your recommendation.

**Commentary on Question**:
*This question tested the real-life problems of the derivative pricing. Most candidates were able earn almost all the points for this question. Candidates were able to correctly identify the issues of using Black implied volatilities in the current interest rate environment and the benefits of Absolute.*

## 8. Continued

*Many candidates struggled in identify the impact and magnitude of the correlations for Black and Absolute volatilities. Most candidates recommended the appropriate volatility and gave a valid justification for its use.*

(i) The long-standing convention of quoting derivative prices for interest rate options like caps, floors, and swaptions is to use Black's formula for option pricing which assumes a lognormal distribution for interest rates.

As rates fell in response to the financial crisis in 2008, Black volatilities changed. Since 2009 there appears to have been three distinctive volatility regimes. In each circumstance, the reason for the increase in volatility regime was a drop in the underlying level of rates. Similar sensitivities of Black implied volatilities to rates are seen in different markets and for different tenor/maturity combinations.

Ideally, an implied volatility measure used for pricing interest rate derivatives would hold approximately constant and vary in an intuitive way under a range of market conditions or across different ranges of instruments including:

- As strike and forward rates change.
- When maturities and tenors vary.
- For different types of instruments (for example puts and calls, payers and receivers) linked via put-call parity.

Before 2008 when rates were higher, Black implied volatilities performed reasonably well against these criteria. However, since 2009 participants in the swaption markets have increasingly chosen to use absolute/normal implied volatilities. The absolute/normal implied volatilities have been considerably more stable, and only a slight positive correlation between absolute volatilities and rate levels is observed. The more stable behavior of absolute/normal rates has led to a view among many traders of interest rate derivatives that interest rate distributions are closer to normal than lognormal.

(ii)

- Correlations between yield curve levels and Black volatilities are typically strongly negative.
- Correlations between yield curve levels and absolute/normal volatilities are weakly positive.

**8. Continued**

(iii) It is suggested the use of absolute/normal implied volatilities in the current low interest rate environment. This conclusion is based on the research which implies normal/absolute volatiles are more robust across a range of interest rate regimes with respect to:

- Changes in strike and forward rates;
- Different maturities and tenors; and
- Different types of instrument.

(e) Describe two approaches to adapt the lognormal forward diffusion LMM to accommodate negative forward rates.

**Commentary on Question**:
*This question tested the additional methodologies in derivative pricing. Most candidates were able to identify using a displacement adjustment to accommodate the negative forward rates but only some were able to also identify using a Stochastic variance process.*

1. Displaced Diffusion LMM
The forward rates are shifted so that the quantities forward rates +δ are positive. And the rates (forward and zero-coupon rates) are floored at -δ. The shift parameter is therefore easily interpretable and stands for the opposite of the lowest absolute level of the interest rates.

2. LMM+
This model is adapted from the LMM by simultaneously shifting the forward rate diffusion and adding a stochastic variance process. The diffusion process is the same as the Displaced Diffusion LMM, but instead of considering a time-dependent volatility function s(t), a stochastic mean-reversion type Cox-Ingersoll-Ross (CIR) process is used.

## 9. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(3k) Understand and apply multifactor interest rate models.

(3l) Demonstrate an understanding of the issues and approaches to building models that admit negative interest rates.

**Sources:**

QFIQ-130-21: Interest Rate Models – Theory and Practice, Second Edition, Brigo, Damiano and Mercurio, Fabio, 2006, Page 143 – 147

QFIQ-129-21: Negative Interest Rates and Their Technical Consequences, AAE, 12/2016, Page 2

**Commentary on Question:**

*Most of the candidates attempted part (a), (b) and (e). Only a few of candidates score full marks on this question.*

**Solution:**

(a) Derive an expression for $r(t)$.

**Commentary on Question**:

*One third of the candidates did not get this question right. Some candidates solved this question using partial differentiation approach, which was incorrect.*

*When writing the equation for y(t), some candidates used $\sigma$ instead of $\eta$. Partial points was given in this case*

*Full points were also given to candidates who did the integration from s to t, instead of 0 to t.*
*Full points was given to candidates who showed most all the steps.*

From Ito's lemma:

$$d(xe^{at}) = axe^{at}dt + e^{at}dx$$

Multiply both sides by $e^{at}$

$e^{at}dx(t) = -ae^{at}x(t)dt + \sigma e^{at}dW_1(t)$

$e^{at}dx(t) + ae^{at}x(t)dt = \sigma e^{at}dW_1(t)$

$$d(xe^{at}) = \sigma e^{at}dW_1(t)$$

## 9. Continued

Integrate both sides:

$$xe^{au}|_0^t = \int_0^t \sigma e^{au} dW_1(u)$$

$$x(t)e^{at} - x(0)e^{a0} = \sigma \int_0^t e^{au} dW_1(u)$$

$$x(t)e^{at} = \sigma \int_0^t e^{au} dW_1(u)$$

We have:

$$x(t) = \sigma \int_0^t e^{-a(t-u)} dW_1(t) du$$

Similarly for y(t),

$$y(t) = \eta \int_0^t e^{-b(t-u)} dW_2(u)$$

Therefore,

$$r(t) = \sigma \int_0^t e^{-a(t-u)} dW_1(u) + \eta \int_0^t e^{-b(t-u)} dW_2(u) + \varphi(t)$$

(b) Derive expressions for $E\{r(t)\}$ and $Var\{r(t)\}$.

**Commentary on Question**:
*Common mistakes for candidates who used approach Var (x + y) = Var (x) + Var (y) + 2Cov (x,y)*

- *The "2Cov(x,y)" term was missing completely*
- *For some candidates who calculated "Cov(x,y)", coefficient 2 was missing*

*Some candidates made mistake at the integration step (e.g. 2 was missing at the denominator).*

Since $E\left(\int_0^t e^{-a(t-u)} dW_1(u)\right) = 0$, hence

$$E[r(t)] = \varphi(t)$$

$\mathrm{Var}[\mathrm{r(t)}] = \mathrm{E}[\mathrm{r}^2(\mathrm{t})] - \mathrm{E}[\mathrm{r(t)}]^2$

## 9. Continued

$$r^2(t) = \sigma^2 \left(\int_0^t e^{-a(t-u)} dW_1(u)\right)^2 + \eta^2 \left(\int_0^t e^{-b(t-u)} dW_2(u)\right)^2 + \varphi^2(t)$$
$$+ 2\,\varphi(t)\left(\sigma \int_0^t e^{-a(t-u)} dW_1(u) + \eta \int_0^t e^{-b(t-u)} dW_2(u)\right)$$
$$+ 2\left(\sigma \int_0^t e^{-a(t-u)} dW_1(u)\right)\left(\eta \int_0^t e^{-b(t-u)} dW_2(u)\right)$$

$$E[r^2(t)] = \sigma^2 \int_0^t e^{-2a(t-u)} du + \eta^2 \int_0^t e^{-2b(t-u)} du + \varphi^2(t)$$
$$+ 2\eta\sigma \int_0^t e^{-(a+b)(t-u)} \rho du$$
$$E[r^2(t)] = \frac{\sigma^2}{2a}(1 - e^{-2at}) + \frac{\eta^2}{2b}\left(1 - e^{-2bt}\right) + \frac{2\eta\sigma\rho}{a+b}\left(1 - e^{-(a+b)t}\right) + \varphi^2(t)$$

Hence,

$$\mathrm{Var}[r(t)] = \frac{\sigma^2}{2a}(1 - e^{-2at}) + \frac{\eta^2}{2b}\left(1 - e^{-2bt}\right) + \frac{2\eta\sigma\rho}{a+b}\left(1 - e^{-(a+b)t}\right)$$

(c) Explain the steps involved in the calibration process for the G2++ model in practice.

**Commentary on Question**:
*Half of the candidates did not answer this question.*
*Full marks will be given to candidate explained the calibration process and the practical constraints.*

Start with the term structure of zero-coupon bond prices at maturities available in the market

$$P^M(0,T)$$

Compute the implied instantaneous forward rates as

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T}$$

Use the equation provided to optimize the parameters a, b, σ, η, ρ

This would imply fitting the full market instantaneous forward curve

(d) Derive an expression for the risk-neutral probability that the instantaneous rate $r(t)$ is negative.

**9. Continued**

**Commentary on Question**:
*Full marks will be given to candidate if the final equation of Q(r(t)) is shown. Some candidates did not recognize that short rate is Gaussian. Partial marks will be given to candidates who mentioned/showed that short rate is Gaussian, but did not get the correct Q(r(t)).*

The short rate is Gaussian with mean $\mu_r(t)$ and variance $\sigma_r^2(t)$, where:

$$\mu_r(t) = E\{r(t)\} = \varphi(t)$$

$$= f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \frac{\eta^2}{2b^2}\left(1-e^{-bt}\right)^2 + \rho\frac{\sigma\eta}{ab}(1-e^{-at})\left(1-e^{-bt}\right)$$

$$\sigma_r^2(t) = \mathrm{Var}\{r(t)\} = \frac{\sigma^2}{2a}(1-e^{-2at}) + \frac{\eta^2}{2b}\left(1-e^{-2bt}\right) + \frac{2\eta\sigma\rho}{a+b}\left(1-e^{-(a+b)t}\right)$$

$$Q\{r(t)<0\} = \Phi\left(\frac{\left(0-\mu_r(t)\right)}{\sigma_r(t)}\right) = \Phi\left(\frac{-\mu_r(t)}{\sigma_r(t)}\right)$$

(e) Comment on the pros and cons of using the G2++ model

**Commentary on Question**:
*Candidate will receive full marks if a pro and a con is provided.*

Pros:

- G2++ model assumes normal distribution for the instantaneous rate and hence can support negative rates
- G2++ model can reproduce ATM volatilities well

Cons:

- G2++ model is not fully satisfactory when attempting to replicate out-of-the-money volatility
- Estimation of parameters is time consuming

(f) Comment on how appropriate this model would be for modeling guarantees that are significantly out of the money.

**Commentary on Question**:
*Partial marks will be given if candidate mentioned that this model is inappropriate.*

While G2++ model can reproduce ATM volatilities well but does not appear to be fully satisfactory when attempting to replicate out-of-the-money volatilities. Thus, the model may not be appropriate for deep out of the money products.

**10.** **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Understand and apply various one-factor interest rate models.

**Sources:**

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro

**Commentary on Question:**

*Overall, candidates performed as expected on this question. There was a mistake in the given equation for A(t;T). However, credit was given when it was due; candidates were not penalized for using the correct or incorrect version equation. The model solution showed the work assuming candidates used the equation for A(t;T) given in the question.*

**Solution:**

(a) Compare $m(r,t)$ with an arbitrage-free parameter $m^*(r,t)$ and explain the meaning of the parameters when $m^*(r,t) = \gamma^*(\bar{r}^* - r)$.

**Commentary on Question**:

*Partial credit was awarded for candidates who demonstrated some understandings with regard to risk-neutral vs. real world as well as mean reversion.*

The drift $m^*(r,t)$ provides arbitrage-free bond return, while $m(r,t)$ does not.

Vasicek model assumes that $m^*(r,t)$ has the same form as the drift rate of the original interest rate process:

$$m^*(r,t) = \gamma^*(\bar{r}^* - r)$$

where $\gamma^*, \bar{r}^*$ are two constants, which $\gamma^*$ controls the sensitivity of the long-term bond prices to variation in the short-term rates.

**10. Continued**

(b) Show that $E\left[\frac{dZ/dt}{Z}\right] = E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1-\gamma^*)$ using Ito's lemma.

**Commentary on Question**:
*Partial credit was awarded for candidates who showed the appropriate partial derivatives and applying them using Ito's Lemma.*

$$\frac{\partial Z}{\partial t} = (A' - B'r)Z, \qquad \frac{\partial Z}{\partial r} = -BZ, \qquad \frac{\partial^2 Z}{\partial r^2} = B^2 Z$$

where

$$\frac{\partial B}{\partial t} = B' = -e^{-r^*(T-t)}$$

$$\frac{\partial A}{\partial t} = A' = (1 + B')\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^*}\right) - \frac{\sigma^2 B(t;T)' B(t;T)}{2\gamma^*}$$

By Ito' lemma:

$$dZ = \left(\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r}\gamma^*(\bar{r}^* - r_t) + \frac{\sigma^2}{2}\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}\sigma dX_t$$

Note that $\gamma^* B = 1 + B'$, thus:

$$A' = \gamma^* B\bar{r}^* - \frac{\sigma^2}{2}B - \frac{\sigma^2(\gamma^* B - 1)B(t;T)}{2\gamma^*} = \gamma^* B\bar{r}^* - \frac{\sigma^2}{2\gamma}B - \frac{\sigma^2}{2}B^2 + \frac{\sigma^2 B(t;T)}{2\gamma^*}$$

So:

$$\frac{\partial Z}{\partial t} = \left[B\gamma^*\bar{r}^* - \frac{\sigma^2}{2}B^2 - B'r\right]Z$$

Therefore:

$$\frac{dZ}{Z} = \left[\gamma^* B\bar{r}^* - \frac{\sigma^2}{2}B - \frac{\sigma^2}{2}B^2 + \frac{\sigma^2 B}{2\gamma^*} - B'r_t - B\gamma^*(\bar{r}^* - r_t) + \frac{\sigma^2}{2}B^2\right]dt + \frac{1}{Z}\frac{\partial Z}{\partial r}\sigma dX_t$$

$$E\left[\frac{\frac{dZ}{dt}}{Z}\right] = \left[-B'E(r_t) + B\gamma^* E(r_t) - \frac{\sigma^2}{2}B + \frac{\sigma^2 B}{2\gamma^*}\right]$$

$$= (-B' + B\gamma^*)E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1-\gamma^*) = E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1-\gamma^*)$$

# 10. Continued

(c) Compute $E\left[\frac{dZ/dt}{Z}\right]$ on zero-coupon bond with 10 years to maturity.

**Commentary on Question**:

*The zero-coupon bond prices given were not correct for the given risk-neutral parameters. However, credit was given where it was due.*

$$B(0{,}10) = \frac{1}{0.4653}(1 - e^{-0.4653*10}) = 2.129$$

$$\mathrm{E}[\mathrm{r}_0] = \mathrm{r}_0 = 2\%$$

Using the result from part (b), we have:

$$E\left[\frac{dZ/dt}{Z}\right] = 2\% + \frac{2.21\%^2 \times 2.129}{2 \times 0.4653}(1 - 0.4653) = 2.0597\ \%$$

(d) Calculate the value of a call option with 1 year to maturity $(T_0 = 1)$, strike price $K = 0.9$, written on a zero-coupon bond with 5 years to maturity.

Under the Vasicek model, a European call open with strike price K and maturity $T_0$ on a zero coupon maturiing on $T_B > T_0$is given by:

$$V(r_0, 0) = Z(r_0, 0; T_B)N(d_1) - KZ(r_0, 0; T_0)N(d_2)$$
$$d_1 = \frac{1}{S}log\left(\frac{Z(r_0, 0; T_B)}{KZ(r_0, 0; T_0)}\right) + \frac{S}{2}$$
$$d_2 = d_1 - S$$
$$S = B(T_0; T_B) * \sqrt{\frac{\sigma^2}{2\gamma^*}(1 - e^{-2\gamma^* T_0})}$$

Thus, we have:

$$B(1; 5) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(5-1)}\right) = 1.815$$
$$S(1{,}5) = 1.815 * \sqrt{\frac{2.21\%^2}{2 * 0.4653}(1 - e^{-2*0.4653})} = 0.03236$$
$$d_1 = \frac{1}{0.03236}log\left(\frac{0.898}{0.9 * 0.975}\right) + \frac{0.03236}{2} = 0.7298$$
$$d_2 = 0.7298 - 0.03236 = 0.6975$$

The value of the call option is:

$$V = 0.898 * N(d_1) - 0.9 * 0.975 * N(0.6975) = 0.02425$$

**11.** **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

**Learning Outcomes:**

(1d) Understand and apply Ito's Lemma.

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

(4l) Explain various issues and approaches for fitting a volatility surface.

**Sources:**

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016, Chapter 19

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Chapter 10, Chapter 12

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

**Commentary on Question:**

*Commentary listed underneath each question component.*

**Solution:**

(a)

(i) Derive $d\Pi_t$ in terms of $dt, dS_t,$ and $dv_t$.

(ii) Construct a riskless portfolio $\Pi_t$ to show that

**11.** **Continued**

$$\frac{F(V_t)-rV_t+rS_t\dfrac{\partial V_t}{\partial S_t}}{\dfrac{\partial V_t}{\partial v_t}}=\frac{F(U_t)-rU_t+rS_t\dfrac{\partial U_t}{\partial S_t}}{\dfrac{\partial U_t}{\partial v_t}}$$

where $F(X_t)=\frac{\partial X_t}{\partial t}+\frac{1}{2}v_tS_t^2\frac{\partial^2 X_t}{\partial S_t^2}+\frac{1}{2}\gamma^2 v_t\frac{\partial^2 X_t}{\partial v_t^2}+\gamma v_t\rho S_t\frac{\partial^2 X_t}{\partial S_t\partial v_t}$ for any derivative $X_t$ on $S_t$.

**Commentary on Question**:
*Candidates performed below expectation on this part. Most candidates attempted part (a)(i), but many did not consider $v_t$ at all or missed the cross term. In part (a)(ii), partial credits were awarded for correctly identifying the required conditions of a riskless portfolio and the dynamic it follows.*

**Part (i)**

$d\Pi_t=\zeta dU_t+\eta dV_t+\theta dS_t$

By Ito's Lemma, we can derive $dV_t$ in terms of $dt, dS_t$, and $dv_t$:

$$dV_t=\frac{\partial V_t}{\partial t}dt+\frac{\partial V_t}{\partial S_t}dS_t+\frac{\partial V_t}{\partial v_t}dv_t$$
$$+\frac{1}{2}v_tS_t^2\frac{\partial^2 V_t}{\partial S_t^2}dt+\frac{1}{2}\gamma^2 v_t\frac{\partial^2 V_t}{\partial v_t^2}dt+\gamma v_t\rho S_t\frac{\partial^2 V_t}{\partial S_t\partial v_t}dt$$

$dU_t$ has the same equation as above except that $V_t$ is replaced by $U_t$.

Plug $dV_t$ and $dU_t$ into $d\Pi_t=\zeta dU_t+\eta dV_t+\theta dS_t$, we obtain

$$d\Pi_t=\left[\eta\left(\frac{\partial V_t}{\partial t}+\frac{1}{2}v_tS_t^2\frac{\partial^2 V_t}{\partial S_t^2}+\frac{1}{2}\gamma^2 v_t\frac{\partial^2 V_t}{\partial v_t^2}+\gamma v_t\rho S_t\frac{\partial^2 V_t}{\partial S_t\partial v_t}\right)\right.$$
$$\left.+\zeta\left(\frac{\partial U_t}{\partial t}+\frac{1}{2}v_tS_t^2\frac{\partial^2 U_t}{\partial S_t^2}+\frac{1}{2}\gamma^2 v_t\frac{\partial^2 U_t}{\partial v_t^2}+\gamma v_t\rho S_t\frac{\partial^2 U_t}{\partial S_t\partial v_t}\right)\right]dt$$
$$+\left(\eta\frac{\partial V_t}{\partial S_t}+\zeta\frac{\partial U_t}{\partial S_t}+\theta\right)dS_t+\left(\eta\frac{\partial V_t}{\partial v_t}+\zeta\frac{\partial U_t}{\partial v_t}\right)dv_t$$

## 11. Continued

**Part (ii)**

Rewrite $d\Pi_t$ as

$$d\Pi_t = [\eta F(V_t) + \zeta F(U_t)]dt + \left(\eta \frac{\partial V_t}{\partial S_t} + \zeta \frac{\partial U_t}{\partial S_t} + \theta\right) dS_t + \left(\eta \frac{\partial V_t}{\partial v_t} + \zeta \frac{\partial U_t}{\partial v_t}\right) dv_t$$

For a riskless portfolio, the coefficient of $dS_t$ and $dv_t$ must be 0, i.e.

$$\begin{cases} \eta \dfrac{\partial V_t}{\partial S_t} + \zeta \dfrac{\partial U_t}{\partial S_t} + \theta = 0 \\ \eta \dfrac{\partial V_t}{\partial v_t} + \zeta \dfrac{\partial U_t}{\partial v_t} = 0 \end{cases} \Rightarrow \begin{cases} \zeta = -\eta \dfrac{\partial V_t / \partial v_t}{\partial U_t / \partial v_t} \\ \theta = -\eta \dfrac{\partial V_t}{\partial S_t} - \zeta \dfrac{\partial U_t}{\partial S_t} \end{cases}$$

The riskless portfolio $\Pi_t$ follows the dynamic $d\Pi_t = r\Pi dt$. Therefore,

$$d\Pi_t = [\eta F(V_t) + \zeta F(U_t)]dt = r\Pi dt = r(\zeta U_t + \eta V_t + \theta S_t)dt$$

By comparing the coefficients of $dt$, it follows that

$$\eta F(V_t) + \zeta F(U_t) = r(\zeta U_t + \eta V_t + \theta S_t)$$

Substitute in $\theta$ and rearrange the terms

$$\eta F(V_t) + \zeta F(U_t) = r\left(\zeta U_t + \eta V_t - \eta S_t \frac{\partial V_t}{\partial S_t} - \zeta S_t \frac{\partial U_t}{\partial S_t}\right)$$

$$\eta\left[F(V_t) - rV_t + rS_t \frac{\partial V_t}{\partial S_t}\right] = -\zeta\left[F(U_t) - rU_t + rS_t \frac{\partial U_t}{\partial S_t}\right]$$

Substitute in $\zeta = -\eta \frac{\partial V_t/\partial v_t}{\partial U_t/\partial v_t}$ and rearrange the terms

$$\frac{F(V_t) - rV_t + rS_t \frac{\partial V_t}{\partial S_t}}{\partial V_t / \partial v_t} = \frac{F(U_t) - rU_t + rS_t \frac{\partial U_t}{\partial S_t}}{\partial U_t / \partial v_t}$$

(b) Derive the generic partial differential equation that any derivative $V_t$ on $S_t$ must follow.

**Commentary on Question**:
*Candidates performed as expected on this part. No credits were awarded for writing down the Black-Scholes equation with a constant volatility.*

## 11. **Continued**

The generic partial differential equation is

$$F(V_t) - rV_t + rS_t\frac{\partial V_t}{\partial S_t} - f(S_t, v_t, t)\frac{\partial V_t}{\partial v_t} = 0$$

Where $F(V_t) = \frac{\partial V_t}{\partial t} + \frac{1}{2}v_t S_t^2\frac{\partial^2 V_t}{\partial S_t^2} + \frac{1}{2}\gamma^2 v_t\frac{\partial^2 V_t}{\partial v_t^2} + \gamma v_t \rho S_t\frac{\partial^2 V_t}{\partial S_t \partial v_t}$.

*Additional explanation: In part (a)(ii), we have derived an equation where all terms concerning $V_t$ are on the left side of the equation, and all terms concerning $U_t$ are on the right side of the equation. It follows that the value of $\left[F(V_t) - rV_t + rS_t\frac{\partial V_t}{\partial S_t}\right]/\frac{\partial V_t}{\partial v_t}$ must not depend on the derivative $V_t$ and is the same for all derivatives on $S_t$.*

(c) Calculate the simulated value of $S_t$ at $t = 0.04$.

**Commentary on Question**:
*Candidates performed poorly on this part. About half of the candidates attempted this part. Many did not calculate the correlated normal random variable $z_3$ or missed multiplying $\sqrt{\Delta t}$ for the Wiener process. Full credits were awarded if candidates chose to use the formula $z_3 = \rho z_1 - \sqrt{1-\rho^2}z_2$, or if they swapped $z_1$ and $z_2$ throughout this part.*

Because $W_t^1$ and $W_t^2$ are two Wiener processes correlated by $\rho$, we need to simulate another normal random variable $z_3$ using the two independent normal random variables so that the correlation between $z_1$ and $z_3$ is $\rho$:

$$z_3 = \rho z_1 + \sqrt{1-\rho^2}z_2 = -0.7 \times 0.1 + \sqrt{1 - 0.1^2} \times 0.9 = 0.8255$$

The increment of Wiener process follows the distribution $W_t - W_0 \sim N(0, t)$.

$v_{0.04}$ and $S_{0.04}$ can be simulated as follows:

$$v_{0.04} = v_0 + k(\theta^2 - v_0)\Delta t + \gamma\sqrt{v_0}\sqrt{\Delta t}\, z_1$$
$$= 0.09 + 0.7 \times (0.16 - 0.09) \times 0.04 + 0.3 \times \sqrt{0.09} \times \sqrt{0.04} \times (-0.7)$$
$$= 0.07936$$

$$S_{0.04} = S_0 + S_0 \times \left(r\Delta t + \sqrt{v_{0.04}}\sqrt{\Delta t}\, z_3\right)$$
$$= 100 + 100 \times \left(0 \times 0.04 + \sqrt{0.07936} \times \sqrt{0.04} \times 0.8255\right)$$
$$= 104.651$$

## 11. Continued

(d) Describe how to produce the volatility smile implied by this Heston model.

**Commentary on Question**:
*Many candidates did not attempt this part. Instead of describing how one can produce the volatility smile, some explained why the Heston model can produce a volatility smile and received no credits.*

1. Simulate $N$ paths of $S_t$ from time 0 to $T$ using the calculation in part (b).
2. Calculate call and put prices at various strikes using the simulated stock price.
3. Plug these calculated prices into the Black-Scholes formula and back-solve for the implied volatility.
4. Plot the implied volatility against the respective strikes to generate the volatility smile at the maturity $T$.

(e)

(i) Explain why the Heston model would produce a volatility smile.

(ii) Describe one way to calibrate the Heston model.

(iii) Describe two potential disadvantages of such calibration.

**Commentary on Question**:
*Many candidates obtained credits on part (i) and part (ii) if they have attempted it. No credits were awarded if the disadvantages described in part (iii) were not specific to the calibration method described in part (ii).*

**Part (i)** Because the volatility produced by the Heston model is stochastic, paths of asset prices ending far from the ATM level will have experienced, on average, a higher volatility than paths ending near ATM. The correlation between the asset price and the volatility naturally generates a volatility smile.

**Part (ii)** The Heston model can be calibrated by minimizing the difference between the volatility generated by the model and the implied volatility in the market.

**Part (iii)** Calibrating a stochastic volatility model to fit the market data potentially have the following disadvantages:

- The calibration can be unstable, resulting in jumps in mark-to-market profit.
- European vanilla option prices cannot be reproduced exactly, so a stochastic volatility model may not be appropriate for vanilla instruments.
- If the model is calibrated to vanilla options, it may not be able to produce the market prices for exotic options, and vice versa.

**12.** **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

**Learning Outcomes:**

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4e) Analyze the Greeks of common option strategies.

**Sources:**

Pricing and Hedging of Financial Derivatives, Ch 6, by Marroni and Perdomo
The Volatility Smile, Derman, Miller, and Park, 2016, Ch. 7

**Commentary on Question:**

*This question tests candidates' understanding of option Greeks.*

**Solution:**

(a) Determine which Greek (Delta, Gamma, Vega, Rho, or Theta) Exhibit I represents. Justify your answer. (Here Theta is defined as the derivative of the option value with respect to the passage of time.)

**Commentary on Question**:

*Candidates performed as expected.*

Exhibit I shows Rho because:
i. Delta is bounded by 1;
ii. Gamma and Vega exhibit bell-shape around at-the-money stock price of $100;
iii. Theta is negative;
Since none of the above pattern fits Exhibit I, it is Rho.

(b) Draw "Line A" in Exhibit I to show the same Greek of a European put option that has the same parameters as the one in Exhibit I. Indicate the Greek value in "Line A" at stock price = 100. You need not show other values in "Line A" but comment on the slope of this line.

**Commentary on Question**:

*Candidates performed below expectations on this part. Partial credit was given when a candidate's answer to part (b) is consistent with the answer to part (a), even though the answer to part (a) is incorrect.*

**12. Continued**

Line A (blue line)

$Call\ Rho = K(T-t)e^{-r(T-t)}N(d_2)$

$Put\ Rho = K(T-t)e^{-r(T-t)}N(-d_2) = K(T-t)e^{-r(T-t)} - \ K(Tle\ -t)e^{-r(T-t)}N(d_2)$

$Put\ Rho = K(T-t)e^{-r(T-t)} - Call\ Rho$

Since $K(T-t)e^{-r(T-t)}$ is a constant, the shape of the Put Rho is same as the Call Rho, but with an oppisite (negative) slope.

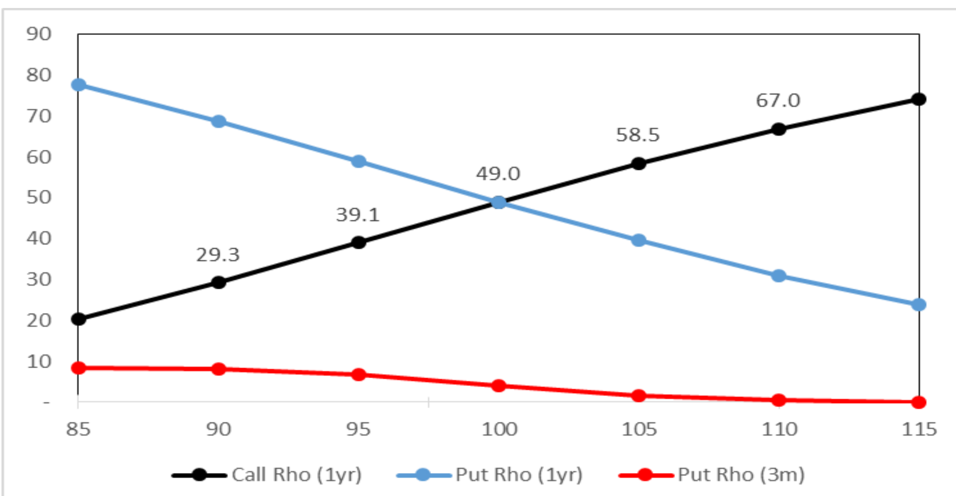
At stock price = 100 = strike price

$$d_2 = \frac{ln\frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \frac{(2\% - \frac{20\%^2}{2})}{20\%} = 0$$

$N(d_2) = N(-d_2) = 0.5$

$Put\ Rho = Call\ Rho = 49.0$

Line A intersects with "Call Rho" at stock price = 100




(c) Draw "Line B" in Exhibit I to show the same Greek of a European put option that has the same parameters as in Exhibit I, except that the time-to-maturity is 1 month. Indicate the Greek value in "Line B" at stock price = 85. You need not show other values in "Line B" but comment on the slope of this line.

**Commentary on Question**:
*Candidates performed below expectations on this part. Partial credit was given when a candidate's answer to part (b) is consistent with the answer to part (a), even though the answer to part (a) is incorrect.*

## 12. Continued

Line B (red line)
For 1 month maturity, at stock price =85:

$$d_2 = \frac{ln\frac{85}{100} + \left(2\% - \frac{20\%^2}{2}\right)(1/12)}{20\%\sqrt{1/12}} = -0.8126$$

$N(-d_2) = 0.7918$

$$\text{1 month Put Rho} = 100 * \left(\frac{1}{12}\right) * e^{-2\%\frac{1}{12}} * 0.7918 = 8.30$$

"Line B" starts at below the "Call Rho" line with a negative slope. For ease of reference, "Line B" is shown in part (b).

(d) Exhibit II below shows Vega and Gamma for a European option on a non-dividend-paying stock. These Greek values are derived from the BSM model with the same strike price, volatility, interest rate, and time-to-maturity as in Exhibit I.

Exhibit II: Vega and Gamma with respect to the underlying stock price

| Stock price | 60 | X |
|---|---|---|
| Vega (shown as the change in the option value to 1 percentage point change of the volatility, e.g., from 25% to 26%) | 0.2401 | 0.2548 |
| Gamma | 0.0267 | 0.0159 |

Determine the stock price X in Exhibit II.

**Commentary on Question**:
*Candidates performed below expectations on this part.*
*Note: The Vega and Gamma values in Exhibit II are derived in the same manner as the Rho in Exhibit I, but they are not based on the option parameters in Exhibit I. Nevertheless, credit was given if X was solved correctly by using the option parameters in Exhibit I.*

$$Vega = 0.01 * S\sqrt{T-t}N'(d_1)$$

$$Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$\frac{Vega}{Gamma} = 0.01 * S^2\sigma(T-t) \qquad (1)$$

**12. Continued**

$$\frac{0.2401}{0.0267} = 0.01 * 60^2 \sigma(T-t) \qquad (2)$$

$$\frac{0.2548}{0.0159} = 0.01 * X^2 \sigma(T-t) \qquad (3)$$

$$\textit{Use equations } (2) \textit{ and } (3) \textit{ to get } X = 60 * \sqrt{\frac{0.2548}{0.0159} * \frac{0.0267}{0.2401}} = 80$$

(e) Determine an upper bound of the option's implied volatility.

**Commentary on Question**:
*Few candidates attempted to answer this part.*

Based on equation (1) from part (d):

$$\sigma = \frac{100 * Vega}{Gamma * S^2(T-t)} \leq \frac{100 * Vega}{Gamma * S^2} \qquad \textit{because } (T-t) > 1$$

$$\sigma \leq \frac{100 * 02401}{0.0267 * 60^2} = 25\%$$

**13. Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

**Learning Outcomes:**

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4b) Identify limitations of the Black-Scholes-Merton pricing formula

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4f) Appreciate how hedge strategies may go awry.

**Sources:**
The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

**Commentary on Question:**
*Commentary listed underneath each question component.*

**Solution:**

(a) Derive the replicating portfolio using options for the interest credited above the guaranteed rate, i.e. $\textit{Interest Credited}_t - g$. Specify each option, including position, option type, term, and strike ratio $K / S_{t-1}$.

**Commentary on Question**:
*Points are awarded for both deriving the formula and correct description of the replication portfolio. Detailed description of the portfolio is required, including, strike ratio, option term, option type.*

$$Interest\ Credited_t - Guar$$
$$= max\left\{min\left[\left(\frac{S_t}{S_{t-1}} - 1\right) * Par, Cap\right], Guar\right\} - Guar,$$
$$= par * max\left\{min\left[\left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Guar}{par}\right), \frac{Cap}{par} - \frac{Guar}{par}\right], 0\right\},$$

**13.** **Continued**

$$= par * max\left\{min\left[\begin{array}{c}\left(\left(\frac{S_t}{S_{t-1}}-1\right)-\frac{Guar}{par}\right),\\ \left(\left(\frac{S_t}{S_{t-1}}-1\right)-\frac{Guar}{par}\right)-\left(\left(\frac{S_t}{S_{t-1}}-1\right)-\frac{Cap}{par}\right)\end{array}\right],0\right\}$$

Denote $G = \left(\left(\frac{S_t}{S_{t-1}}-1\right)-\frac{Guar}{par}\right)$, $C = \left(\left(\frac{S_t}{S_{t-1}}-1\right)-\frac{Cap}{par}\right)$,
as $Cap > Guar$, $G > C$.

Thus, $Interest\ Credited_t - Guar$
$= par * max[min(G, G - C), 0]$
$= par * \{max[0, -min(G, G - C)] + min(G, G - C)\}$
$= par * \{max[0, max(-G, C - G)] - max(-G, C - G)\}$
$= par * \{max[G, max(0, C)] - max(0, C)\}$
$= par * \{max[0, max(G, C)] - max(0, C)\}$
$= par * \{max(0, G) - max(0, C)\}$, as $G > C$
$= par * \left\{max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Guar}{par}\right), 0\right] - max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Cap}{par}\right), 0\right]\right\}$.

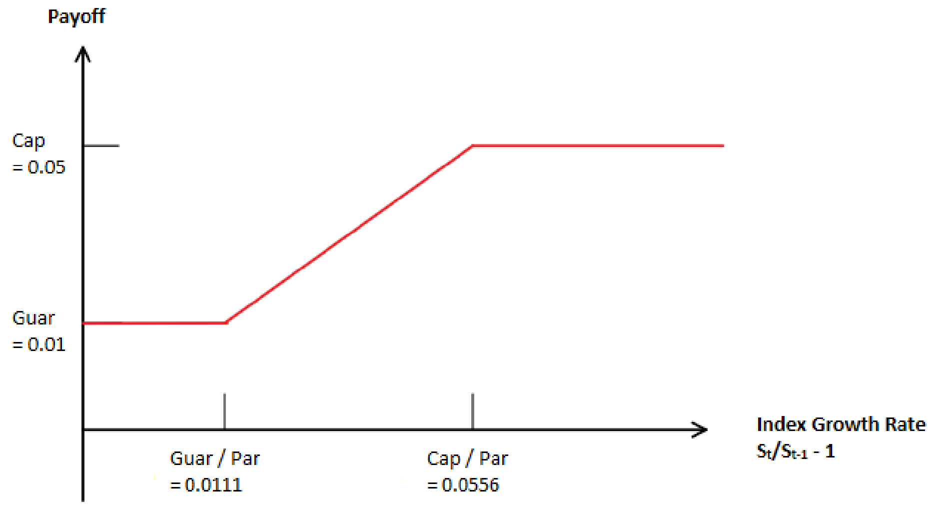
Therefore, the interest credited above the guaranteed rate can be replicated by $p$ units of call spread, with the following options:

- Long position of a 1-year term European call option, with strike ratio $\frac{K_L}{S_{t-1}} = 1 + \frac{Guar}{par} = 1 + \frac{0.01}{0.9} = 1.0111$
- Short position of a 1-year term European call option, with strike ratio $\frac{K_S}{S_{t-1}} = 1 + \frac{Cap}{par} = 1 + \frac{0.05}{0.9} = 1.0556$

(b) Sketch the payoff of the replicating portfolio against the index growth rate $\left(\frac{S_t}{S_{t-1}} - 1\right)$.

**Commentary on Question**:
*Correct shape of the curve as well as identification of both the turning points are required for full credit.*

**13. Continued**




Turning point:

$$\left(\frac{Guar}{Par}, Guar\right) = (0.0111, 0.01)$$

and

$$\left(\frac{Cap}{Par}, Cap\right) = (0.0556, 0.05).$$

(c)

(i) Calculate the interest credited on Dec 31, 2019.

(ii) Calculate the cost of the replicating portfolio for the interest credited above the guaranteed rate on Dec 31, 2018.

**Commentary on Question**:
*For part (i), interest percentage as well as dollar amount need to be specified. For part (ii), each step needs to be shown clearly and demonstrate how each parameter is calculated.*

(i)

$$\begin{aligned}
&Interest\ Credited_t \\
&= max\left\{min\left[\left(\frac{S_t}{S_{t-1}} - 1\right) * Par, Cap\right], Guar\right\}, \\
&= max\left\{min\left[\left(\frac{1080}{1000} - 1\right) * 90\%, 5\%\right], 1\%\right\}, \\
&= max\{min[(7.2\%, 5\%)], 1\%\} \\
&= 5\%.
\end{aligned}$$

**13. Continued**

Therefore, interest credited per \$1000 of investment $= 5\% * 1000 = \$50$.

(ii)

From (a), the interest crediting strategy can be replicated by the following call spread:
$par * \left\{max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Guar}{par}\right), 0\right] - max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Cap}{par}\right), 0\right]\right\}$.

The cost of the replicating portfolio is the option value of this call spread at $t - 1$.

Using Black-Scholes model to calculate the option value,
$C(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$,

where
$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)\right]$,
$d_2 = d_1 - \sigma\sqrt{T - t}$.

Option value of the long position of a 1-year term European call option with strike ratio $K_L/S_{t-1} = 1.0111$ is:

$C_L(S_{t-1}, t - 1) = S_{t-1} N(d_1) - K_L e^{-r} N(d_2)$,

where
$K_L = S_{t-1}\left(1 + \frac{Guar}{par}\right) = 1000 \; * 1.0111 = 1011.11$
$d_1 = \frac{1}{\sigma}\left[\ln\left(\frac{1}{1+\frac{Guar}{par}}\right) + \left(r + \frac{\sigma^2}{2}\right)\right] = \frac{1}{0.2}\left[\ln\left(\frac{1}{1+\frac{0.01}{0.9}}\right) + \left(0.05 + \frac{0.2^2}{2}\right)\right] = 0.2948$
$N(d_1) = 0.6141$.
$d_2 = d_1 - \sigma = 0.2948 - 0.2 = 0.0948$, $N(d_2) = 0.5359$.
$C_L(S_{t-1}, t - 1) = 1000 * 0.6141 - 1011.11 * e^{-0.05} * 0.5359 = \$98.7$.

Option value of the short position of a 1-year term European call option with strike ratio $K_S/S_{t-1} = 1.0556$ is:

$C_S(S_{t-1}, t - 1) = S_{t-1} N(d_1) - K_S e^{-r} N(d_2)$,

where
$K_S = S_{t-1}\left(1 + \frac{Cap}{par}\right) = 1000 \; * 1.0556 = 1055.56$.
$d_1 = \frac{1}{\sigma}\left[\ln\left(\frac{1}{1+\frac{Cap}{par}}\right) + \left(r + \frac{\sigma^2}{2}\right)\right] = \frac{1}{0.2}\left[\ln\left(\frac{1}{1+\frac{0.05}{0.9}}\right) + \left(0.05 + \frac{0.2^2}{2}\right)\right] = 0.0797$,
$N(d_1) = 0.5319$.
$d_2 = d_1 - \sigma = 0.0797 - 0.2 = -0.1203$,

## 13. Continued

$N(d_2) = 1 - N(-d_2) = 1 - 0.5478 = 0.4522.$
$C_S(S_{t-1}, t-1) = 1000 * 0.5319 - 1055.56 * e^{-0.05} * 0.4522 = \$77.8.$

Therefore, the total cost = $par * [C_L(S_{t-1}, t-1) - C_S(S_{t-1}, t-1)] = 0.9 * (\$98.7 - \$77.8) = \$18.8.$

(d)

(i) Calculate the effective volatility $\tilde{\sigma}$ that covers the transaction costs for long and short option positions, respectively. Assume 52 weeks per year and $\pi = 3.14$.

(ii) Justify the calculation of effective volatility regarding to each option position.

**Commentary on Question**:

*For part (i), solutions using the variance formula* $\sigma^2 \pm 2\sigma k\sqrt{\frac{2}{\pi dt}}$ *are awarded full credit as well.*

(i)

The effective volatility $\tilde{\sigma}$ for long call position = $\sigma - k\sqrt{\frac{2}{\pi dt}} = 20\% - 0.52\% \times \sqrt{\frac{2}{3.14} \times \frac{52}{1}} = 17.00\%.$

The effective volatility $\tilde{\sigma}$ for short call position = $\sigma + k\sqrt{\frac{2}{\pi dt}} = 20\% + 0.52\% \times \sqrt{\frac{2}{3.14} \times \frac{52}{1}} = 23.00\%.$

(ii)

When you long an option, you should pay less than the fair BSM value, since the hedging cost will diminish your P&L. Fora long position, the effective volatility is reduced.

When you short an option, you must ask for more money to cover your hedging costs, and therefore you should have sold it for a greater price than the BSM value. For a short position, the effective volatility should be enhanced.

(e)

(i) Describe the relationship between hedging frequency and the profit.

(ii) Describe strategies that can be used for rebalancing.

## 13. **Continued**

**Commentary on Question**:
*For part (i), the candidate needs to mention smaller hedging error leads to more certainty regarding the profit. "Frequent rebalancing reduces hedging error" does not answer the question.*
*For part (ii), reasonable description of benchmarks that trigger rebalancing are accepted.*

(i)

The more you rebalance:

- the smaller the hedging error, the more certain about the profit,

- but the greater the cost and the smaller the expected profit as the more of profit is given away in transaction costs.

(ii)

Rebalancing strategies:

- Rebalancing at regular intervals: set a time interval and rebalance at the end of every time step, no matter how little or how much additional options must be traded.

- Rebalancing Triggered by changes in the hedge ratio: set a trigger rate and rebalance only after a substantial change in the hedge ratio has occurred, where the trigger rate is hit.

**14.** **Learning Objectives:**

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

**Learning Outcomes:**

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
  (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
  (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

**Sources:**

QFIQ-122-20: Equity Indexed Annuities: Downside Protection, But at What Cost?

QFIQ-132-21: Investment Instruments with Volatility Target Mechanism, Albeverio, Steblovskaya, and Wallbaum, 2013

**Commentary on Question:**

*Commentary listed underneath question component.*

**Solution:**

(a) (i)

The potential product has the following features:

- Minimum Guarantee or floor value on the contract, as the cumulative return has a lower bound. The client is guaranteed a minimum value.
- Index Participation rate: The client shares in the growth of the reference portfolio (or index). The client's returns are a percentage of the reference portfolio (i.e., 70%). The slope is less steep than that of the S&P 500.
- Growth Rate Cap: A cap on the maximum crediting rate for the applicable period. The cumulative return has an upper bound.
- Surrender Charge period

(a)(ii)

The buyer of the potential product has discretionary long-term wealth and exhibits a great sensitivity to down-side risk. The floor value is attractive to someone who is sensitive to changes in wealth as it approaches the critical floor.

## 14. Continued

The surrender charges mean that the product is not liquid, so it cannot be used as precautionary savings for economic downturns (discretionary long-term wealth).

(b) Describe a condition when the market-value adjustment works in an investor's favor.

**Commentary on Question**:
*Candidate did well on this question, but quite a few did not specify that MVA would work in an investor's favor when the current market rate drops below the declared rate.*

The market-value adjustment works in an investor's favor when the contract is surrendered in a low-interest rate environment, where the current market rate drops below the declared interest rate.

(c) Calculate the risk budget of the EIA contract.

**Commentary on Question**:
*Most candidates understood the calculation of risk budget, but missed correctly factoring the minimum guarantee of 3%.*

At time 0, the insurer received $100,000 initial investment from the policyholder. In order to guarantee minimum 3% interest rate credit on the 100% of the initial investment at maturity of 1-year, the amount 100,000*1.03*$e^{-0.03\times 1}$ needs to be invested at time 0 in a zero-coupon bond (where the 3% is the continuously compounded yield of the 1-year zero-coupon bond). Thus, the remaining part of the initial investment constitutes the risk budget of the guarantee structure, which is calculated as

RB = 100,000 – 103,000*$e^{-0.03\times 1}$
RB = 44

(d) Assess if the risk budget in part (c) is sufficient to fund the option purchase that would replicate the EIA payoff in excess of the minimum return.

**Commentary on Question**:
*Many candidates struggled to calculate the call option price.*

The strike of the call option is 1.03, where 0.03 is the minimum guaranteed interest rate credit. Thus, the call option price factor is .07986. Total cost of the option purchase is 100,000 x .07986 = 7,986, which is higher than the risk budget. Therefore, the risk budget is not sufficient to fund the option purchase that would replicate the EIA payoff in excess of minimum returns.

**14.** **Continued**

(e) Calculate the index participation rate to break-even using the risk budget calculated in part (c).

**Commentary on Question**:
*Most candidates understood the concept of the participate rate.*

Break-even Participation rate = RB/Option price = 44/7,986 = 0.55%

(f) State and explain whether each of the above statements is true or false.

**Commentary on Question**:
*Most candidates received partial credits on this question.*

- Statemant A: False. An EIA provides an insurance company with limited opportunity for either actuarial or investment gains.

- Statement B: True. The participation rate is applied to the index's price appreciation, and not applied to the dividend yield.

- Statement C: False. An EIA is not liquid and has surrender charges, which prevent it from consideration as a precautionary savings vehicle.

**15.** **Learning Objectives:**

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

**Learning Outcomes:**

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:

(i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.

(ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

**Sources:**

Hedging Variable Annuities: How Often Should the Hedging Portfolio be Rebalanced?, Risks and Rewards, Feb 2018

QFIQ-128-20: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

**Commentary on Question:**

*The question is to test the candidates' ability to identify and evaluate embedded options in liabilities specifically GMWB, to demonstrate the understanding of the effectiveness of different hedging strategies, and to explain and evaluate the effectiveness of different hedging strategies through the statistical information provided.*

*The candidates are required to not just provide the results but also give the explanation, especially the connection between the theory and statistical information.*

**Solution:**

(a) Compare using a daily vs. less frequent rebalancing strategy for hedging VA guarantees.

**Commentary on Question**:

*Most candidates only explain the differences in the transaction cost and model efficiency between more frequent and less frequent rebalancing. Some candidates do provide very deeper understanding of the differences outlined below.*

Hedging portfolios can be rebalanced daily or less frequently.

- More frequent rebalancing leads to a more effective hedge under a Black-Scholes setting.
- Empirical Evidence favors less frequent rebalancing of hedging portfolios, as hedging strategies are exposed to model risk.

# 15. Continued

- Larger transaction costs are associated with more frequent rebalancing.
- A Black Scholes delta hedge at larger time scales is generally exposed to less model risk than a daily hedge, as they tend to conform better to the Gaussian hypothesis (Aggregational Gaussianity). Returns on monthly time scales are closer to being normally distributed than daily returns.
  Move based strategies also underperform monthly returns due to this fact, as they require more frequent rebalancing in periods of higher volatility/kurtosis (i.e.. when returns further deviate from normal).
- Annualized volatility of monthly returns is below that of daily and weekly returns, due to negative autocorrelations observed in the daily returns. Negative autocorrelations in daily returns at short lags have been observed. They imply some level of short-term mean reversion which contributes to reducing noise and volatility in aggregated returns.

(b)

(i) Compare the hedging strategies considered above (for the GMWB rider) with respect to the following aspects:

I. Hedging Delta (Δ) versus Delta - Rho (Δ- ρ)
II. Monthly versus daily rebalancing
III. BS versus BSV hedging model

(ii) Recommend which one of the four hedging strategies above that Company JCP should implement, based on part (b)(i).

(iii) Describe how the hedging strategy that you recommended would perform in a persistently low interest rate environment.

**Commentary on Question**:
*Majority of the candidates provide the results but only a few candidates could provide the explanation and the connection of the statistical information and the supporting reasons.*
*Many candidates only mention about CTE rather explain the differences of CTE.*

(i)Adding a Rho / interest rate hedge improves the overall quality of the hedging strategy.  The Rho hedge is not as effective in the tail of the hedged loss distribution, as the improvement is more moderate in the chart when interest rate model risk is introduced. A Rho hedge does not limit the volatility risk, which is the risk reduction from row 1 to row 2 is smaller in model 3.

**15. Continued**

More frequent rebalancing reduces hedging risk.v However, model risk reduces the benefit of hedging more frequently (Improvement in Model 3 is less than Model 1). vComputing hedging positions more frequently with the wrong model could lead to a larger accumulation of hedging errors. More frequent rebalancing also comes with higher monitoring and trading costs.

Modeling interest rates stochastically does not have a definitive impact on hedging risk, as the impact is small and the direction is not consistent between financial market models. The table of hedged loss' results indicates that there is a small impact of hedging using BSV model compared to using BS model. The inclusion of a rho hedge seems to be more important for a hedging program compared with the choice of a model for interest rates.

(ii) I would recommend using a hedging strategy with a $\Delta$- $\rho$ hedge, rebalanced daily with an BS model. The rho hedge and daily rebalancing improved the hedging risk, which the BSV model did not have a definite improvement and added complexity to the model.

(iii)Interest rates are currently very low, so they may either remain low or rise. Guarantee values will decrease if interest rates rise, which will result in a gain for the insurer if interest rate risk is not hedged. Hedging interest rate risk will benefit insurers if interest rates remain low, as they can be exposed to large hedging losses.

Delta-Rho hedging strategies are not significantly impacted by the interest rate environment.

Delta only hedging strategies may result in a gain for the insurer if interest rates rise, but it may result in a loss for the insurer if interest rates remain low.

(c) Define the two schools of thought regarding the calibration of the equity stochastic volatility parameter and their application to VA hedging programs.

**Commentary on Question**:
*Not many candidates could list the two schools of the thought regarding the calibration of the equity stochastic volatility parameter and their application to VA hedging programs.*
*Partial marks are given to those who mention calibration, extract from data or similar arguments.*

**15. Continued**

Backward looking: Calibration based on Historical Data

- Stable estimate over time and preferred approach for most companies

Forward Looking: Extracted from market data, such as implied volatility surface from vanilla options

- Difficult to get the implied volatility as options over 2-3 years to maturity are sold on the over-the-counter market (VA contracts are long)
- Implied volatilities may not produce appropriate volatility inputs for hedging VAs with non-vanilla features (i.e., GMWB)

(d) Calculate the 1-year $VIX_t$ under the following stochastic differential equation:

- $dv_t = 3(2 - v_t)dt + \sigma_v \sqrt{v_t}\, dW_t^v$
- $\lambda_v = -1$
- $\{W_t^v, t \geq 0\}$ is a standard Brownian motion under the real-world measure

**Commentary on Question**:
*Very few candidates try the question. No candidate demonstrates the understanding that the risk neutral dynamics are needed to generate the values of the VIX.*

The risk neutral dynamics are needed to generate the values of the VIX. Let $\lambda_v(t, v_t)$ represent the market price of risk process and assume $\{\widetilde{W}_t^v, t \geq 0\}$ is a risk neutral Brownian motion, where $\widetilde{W}_t^v = W_t^v + \int_0^t \lambda_v(t, v_t)\, ds$. In the Heston model $\lambda_v(t, v_t) = \frac{\lambda_v \sqrt{v_t}}{\sigma_v}$, where $\lambda_v$is the volatility risk premium parameter. The risk neutral dynamics are given by $dv_t = \tilde{\kappa}(\tilde{\theta} - v_t)dt + \sigma_v \sqrt{v_t}\, d\widetilde{W}_t^v$.

$\tilde{\kappa} = \kappa + \lambda_v = 3 - 1 = 2$

$$\tilde{\theta} = \frac{\kappa\theta}{\kappa + \lambda_v} = \frac{3 * 2}{3 - 1} = 3$$

$$A = \tilde{\theta}\left(1 - \frac{1 - e^{-\tilde{\kappa}}}{\tilde{\kappa}}\right) = 3 * \left(1 - \frac{1 - e^{-2}}{2}\right) = 1.703003$$

$$B = \left(\frac{1 - e^{-\tilde{\kappa}}}{\tilde{\kappa}}\right) = \left(\frac{1 - e^{-2}}{2}\right) = 0.4323324$$

$1 - \text{year } VIX_t = \sqrt{A + Bv_t} = \sqrt{1.7 + 0.432v_t}$