SOCIETY OF ACTUARIES

Advanced Short-Term Actuarial Mathematics Exam

Exam ASTAM

Date: October 23, 2024

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 6 questions numbered 1 through 6 with a total of 60 points. The points for each question are indicated at the beginning of the question.

2. **Question 1 is to be answered in the Excel workbook. For this question, only the work in the Excel workbook will be graded.**

3. Questions 2-6 are to be answered in pen in the Yellow Answer Booklet provided. For these questions graders will only look at the work in the Yellow Answer Booklet. Excel may be used for calculations or for statistical functions, but any work in the Excel booklet will not be graded.

4. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

Excel Answer Instructions

1. For Question 1, you should answer directly in the Excel Question worksheet. The question will indicate where to record your answers.
2. You should generally use formulas in Excel rather than entering solutions as hard coded numbers. This will aid graders in assigning appropriate credit for your work.
3. Graders for Question 1 will not have access to any comments or calculations provided in the Yellow Answer Booklet.
4. For Question 1, you may add notes to the Excel Question worksheet if you feel that might help graders. However, these should be entered directly into the Excel Question worksheet. Graders may not be able to read notes entered as comments.
5. When you finish, save your Excel workbook with a filename in the format xxxxx_ASTAM where xxxxx is your candidate number. Your name must not appear in the filename.

Pen and Paper Answer Instructions

1. Write your candidate number and the number of the question you are answering at the top of each sheet. Your name must not appear.

2. Start each question on a fresh sheet. You do not need to start each sub-part of a question on a new sheet.

3. Write in pen on the lined side of the answer sheet.

4. The answer should be confined to the question as set.

5. When you are asked to calculate, show all your work including any applicable formulas in the Yellow Answer Booklet.

6. If you use Excel for calculations for pen and paper answers, you should include as much information in the Yellow Answer Booklet as if you had used a calculator, including formulas and intermediate calculations where relevant. Written answers without sufficient support will not receive full credit.

7. When you finish, hand in <u>all</u> your written answer sheets to the Prometric Center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

8770 W. Bryn Mawr Avenue, Suite 1000
Schaumburg, IL 60631-1516

****BEGINNING OF EXAMINATION****
*****ADVANCED SHORT-TERM ACTUARIAL MATHEMATICS*****

Provide the response for Question 1 in the Excel Question worksheet

1.

(*10 points*) You are fitting a Pareto distribution with parameters α and θ to loss severity data, using maximum likelihood estimation (MLE).

There are $n = 100$ observations.

The sorted individual loss severity values, denoted by y_i, for $i = 1, 2, \ldots, 100$, are given in Column N of the Excel spreadsheet.

You should use the columns to the right of the data for your calculations. Do not insert any columns or rows into the data table shown in columns M and N.
Let $\ln\left(f\left(y\right)\right)$ denote the natural log of the Pareto probability density function.

(a) (*4 points*) Assume first that $\alpha = 4.00$ and $\theta = 13,000$.

(i) Calculate $\ln\left(f\left(y_1\right)\right)$.

(ii) Calculate the total log-likelihood for the sample.

(iii) Calculate 10^5 times the derivative with respect to θ of $\ln\left(f\left(y_1\right)\right)$. Your answer should be -7.3 to the nearest 0.1.

(iv) Calculate 10^5 times the derivative with respect to θ of the total log-likelihood for the sample.

(b) (*3 points*) You are given that the MLE of α is $\hat{\alpha} = 4.440$

(i) Use Goal Seek to determine the MLE of θ.

(ii) Determine the maximum value of the log-likelihood for the fitted distribution. Your answer should be –926.0 to the nearest 0.1.

1. **Continued**

(c) (*3 points*) Your colleague has fitted a 3-parameter translated Pareto distribution to the same data. The maximum log-likelihood for that distribution is –921.8.

(i) Calculate the test statistic for a likelihood ratio test to compare the two fitted distributions.

(ii) Write down the degrees of freedom for this likelihood ratio test.

(iii) Evaluate the p-value for the likelihood ratio test.

(iv) State clearly the conclusion of the likelihood ratio test.

2.

(*7 points*)

(a) (*3 points*) ABC insurance sells automobile collision coverage with ordinary deductibles of 250, 500, and 1,000. The base rate is based on a deductible of 500.

You are given the following data:

Loss Size	**Number of Losses**	**Ground Up Total Loss**
0 – 250	1,000	150,000
251 – 500	2,000	650,000
501 – 1,000	6,000	4,320,000
Over 1,000	20,000	90,000,000
All	29,000	95,120,000

(i) Calculate the indicated deductible relativity for the deductible of 250.

(ii) Calculate the indicated deductible relativity for the deductible of 1,000.

(iii) Explain why the ground-up loss severity distribution might differ for different deductible levels.

ABC also sells liability insurance with limits of 250,000, 1,000,000, and 5,000,000. The base rate is based on a limit of 250,000.

You are given the following data:

Size of Loss	**Number of Claims**	**Ground Up Total Losses (in Millions)**
1 – 250,000	1,200	180.00
250,001 –1,000,000	850	361.25
1,000,001 – 5,000,000	150	187.50
All	2,200	728.75

(b) (*2 points*)

(i) Calculate the increased limit factor (ILF) for the limit of 1,000,000.

(ii) Calculate the increased limit factor (ILF) for the limit of 5,000,000.

(c) (*2 points*) Describe two practical considerations regarding the data to use for determining ILFs.

3.

(*12 points*) An insurance company sells one-year insurance policy on a risk with the following loss frequency distribution

Number of losses	Probability
0	0.5
1	0.3
2	0.1
3	0.1

You are also given the following information.

- The severity of each loss X_j follows a lognormal distribution with parameters $\mu = 7$ and $\sigma = 0.5$.
- The number of losses and the amount of each loss are assumed to be mutually independent.
- The policy has an ordinary deductible of $d = 500$ per loss.
- N^L denotes the number of loss events in a single year.
- N^P denotes the number of payments in a single year.

(a) (*3 points*)

(i) Calculate the expected value of the aggregate ground-up loss.

(ii) Calculate the standard deviation of the aggregate ground-up loss.

(iii) Explain briefly why the Normal Approximation would not be a good method of approximating the distribution of the aggregate ground-up loss for this policy.

(b) (*2 points*) Calculate the loss elimination ratio.

(c) (*2 points*) Let $I_j = \begin{cases} 0 & X_j \le d \\ 1 & X_j > d \end{cases}$ and let $E[I_j] = v$

so that $N^P = \sum_{j=1}^{N^L} I_j$ (where we define $N^P = 0$ when $N^L = 0$).

Prove that the PGF for N^P is

$P_{N^P}(z) = P_{N^L}(1 - v(1 - z))$, where $v = 0.9419$.

3. Continued

(d) (*2 points*)

(i) Calculate the probability that the aggregate payment from the coverage is 0.

(ii) Calculate the probability that there is exactly one payment made in a year.

(e) (*3 points*) The insurer purchases an excess-of-loss reinsurance policy such that the insurer's own maximum payment on an individual claim is 2,000. The reinsurance premium is set at 120% of the expected annual reinsurance claim payments.

(i) Calculate the annual reinsurance premium for this policy.

(ii) The amount of each loss in the next year is expected to increase uniformly by 10% due to inflation, and all contract terms remain the same. As a result, the reinsurance premium for next year is expected to be increased by h.

Without further calculation, state with reasons whether h is less than, equal to, or greater than 10%.

4.

(*10 points*) An insurance company has a portfolio of insurance policies for which the loss random variable, $X > 0$, has the following survival function:

$$S_X(x) = \left(\frac{\lambda}{\lambda + \theta x}\right)^{\frac{1}{\theta}} \qquad \lambda > 0, \theta > 0$$

(a) (*3 points*)

(i) Show that the hazard rate function for *X* is $h(x) = \dfrac{1}{\lambda + \theta x}$.

(ii) Briefly categorize the tail behavior of *X*, based on *h*(*x*).

(b) (*1 point*) The insurance company is deciding whether to use the Value at Risk or Expected Shortfall risk measure to determine its solvency capital needed to cover adverse loss experience.

Describe one advantage and one disadvantage of using the Expected Shortfall risk measure, as compared with the Value at Risk measure.

(c) (*3 points*) Now let $\lambda = 2000$ and $\theta = 0.4$.

(i) Show that for the distribution of *X* the 99% Value at Risk (VaR), denoted $Q_{99\%}$, is 26,550 to the nearest 10. You should calculate the value to the nearest 1.

(ii) You are given that the mean excess loss function for *X* is

$$e(d) = \frac{\lambda + \theta d}{1 - \theta} \text{ for } 0 < \theta < 1.$$

Calculate the confidence level α such that the α-Expected Shortfall, denoted ES_α, is the same as the 99% VaR calculated in (c)(i).

(iii) In general, suppose that $Q_\beta = ES_\alpha$ for some α, β. Explain why it is necessary that $\alpha \le \beta$.

4. **Continued**

(d) (*3 points*)

(i) Identify the functions c_n and d_n (not involving *x*) such that for any *x*, $\theta > 0$,

$$nS(c_n x + d_n) = (1 + \theta x)^{-1/\theta}$$

(ii) Explain the significance of this result in terms of the relationship between θ and the tail behavior of *X*.

5.

(*10 points*)

(a) (*1 point*) An insurer uses the Bühlmann Credibility Model to estimate expected claim frequency for risks in a portfolio. Let X denote the number of claims from an individual risk in a single year.

You are given that $\mu(\Theta) = \mathrm{E}\left[X \mid \Theta\right]$ and that $v(\Theta) = \mathrm{Var}\left[X \mid \Theta\right]$.

Define the Bühlmann Credibility Model parameters $\mu, v,$ and a in terms of the functions $\mu(\Theta)$ and $v(\Theta)$.

The insurer will use the semiparametric approach to estimate the Bühlmann Credibility Model parameters $\mu, v,$ and a.

The insurer assumes that the frequency of claims for an individual policy, conditional on an unknown parameter β, follows a Geometric distribution with parameter β.

(b) (*1 point*)

(i) Show that $\mu = E\left[\beta\right]$

(ii) Show that $a = \mathrm{Var}\left[\beta\right]$

(c) (*3 points*)

(i) Show that $\mathrm{Var}\left[X\right] = 2\mathrm{Var}\left[\beta\right] + \mathrm{E}\left[\beta\right]\left(1 + \mathrm{E}\left[\beta\right]\right)$

(ii) Hence show that

$$a = \frac{\mathrm{Var}\left[X\right] - E\left[X\right]\left(1 + E\left[X\right]\right)}{2}$$

5. **Continued**

(d) (*3 points*) You are given the following annual claims data for the portfolio

Number of Claims in a policy year	Number of Policies
0	2,230
1	209
2	41
3	12
4	8
Total	2,500

(i) Show that $\hat{\mu} = 0.14$ to the nearest 0.01. You should calculate the value to the nearest 0.0001.

(ii) Calculate the standard deviation of the claim frequency for the portfolio.

(iii) Show that $\hat{a} = 0.03$ to the nearest 0.01. You should calculate the value to the nearest 0.001.

(iv) Calculate $\hat{v}$.

(e) (*2 points*)

(i) Calculate the Bühlmann Credibility estimate of the expected claim frequency for a policy which has been in force for 1 year and had 1 claim in that year.

(ii) Calculate the Bühlmann Credibility estimate of the expected claim frequency for a policy that has been in force for 5 years and had 1 claim each year in that period.

(iii) Explain why the expected claim frequency for the policy in (e)(i) is less than the expected claim frequency for the policy in (e)(ii), even though the historical claim frequencies are the same.

6.

(*11 points*) You are analyzing a run-off triangle of aggregate claims. The development factors are:

DY, j	0	1	2	3	4	5	6	7
f_j	3.0982	1.4436	1.1955	1.0874	1.0360	1.0186	1.0056	1.0000

(a) (*2 points*)

(i) State with reasons whether this business is long tailed or short tailed.

(ii) You are given that the most recent cumulative claims data for Accident Year (AY) 6 is $C_{6,1} = 9{,}182$.

Show that the estimated outstanding claims for AY 6 total 9,100 to the nearest 100, using the Chain Ladder method. You should calculate the value to the nearest 1.

(b) (*2 points*)

(i) The Earned Premium in AY 6 is 18,000 and the expected loss ratio at issue is 0.85. Show that the estimated outstanding claims for AY6 using the Bornhuetter-Ferguson method is 7,600 to the nearest 100. You should calculate the value to the nearest 1.

(ii) Describe one advantage and one disadvantage of the Bornhuetter-Ferguson method compared with the Chain Ladder method.

(c) (*3 points*) You are given the following four assumptions for Mack's model.

(A1) $C_{i,j}$ and $C_{l,k}$ are independent for $i \neq l$ and for all j, k.

(A2) For a given accident year (AY) i, $\{C_{i,j}\}_{j=0,1,2...}$ is a Markov Chain.

(A3) There exist f_j such that $\mathrm{E}\left[C_{i,j+1} \middle| C_{i,j}\right] = f_j\ C_{i,j}$.

(A4) There exist σ_j^2 such that $\mathrm{Var}\left[C_{i,j+1} \middle| C_{i,j}\right] = \sigma_j^2\ C_{i,j}$.

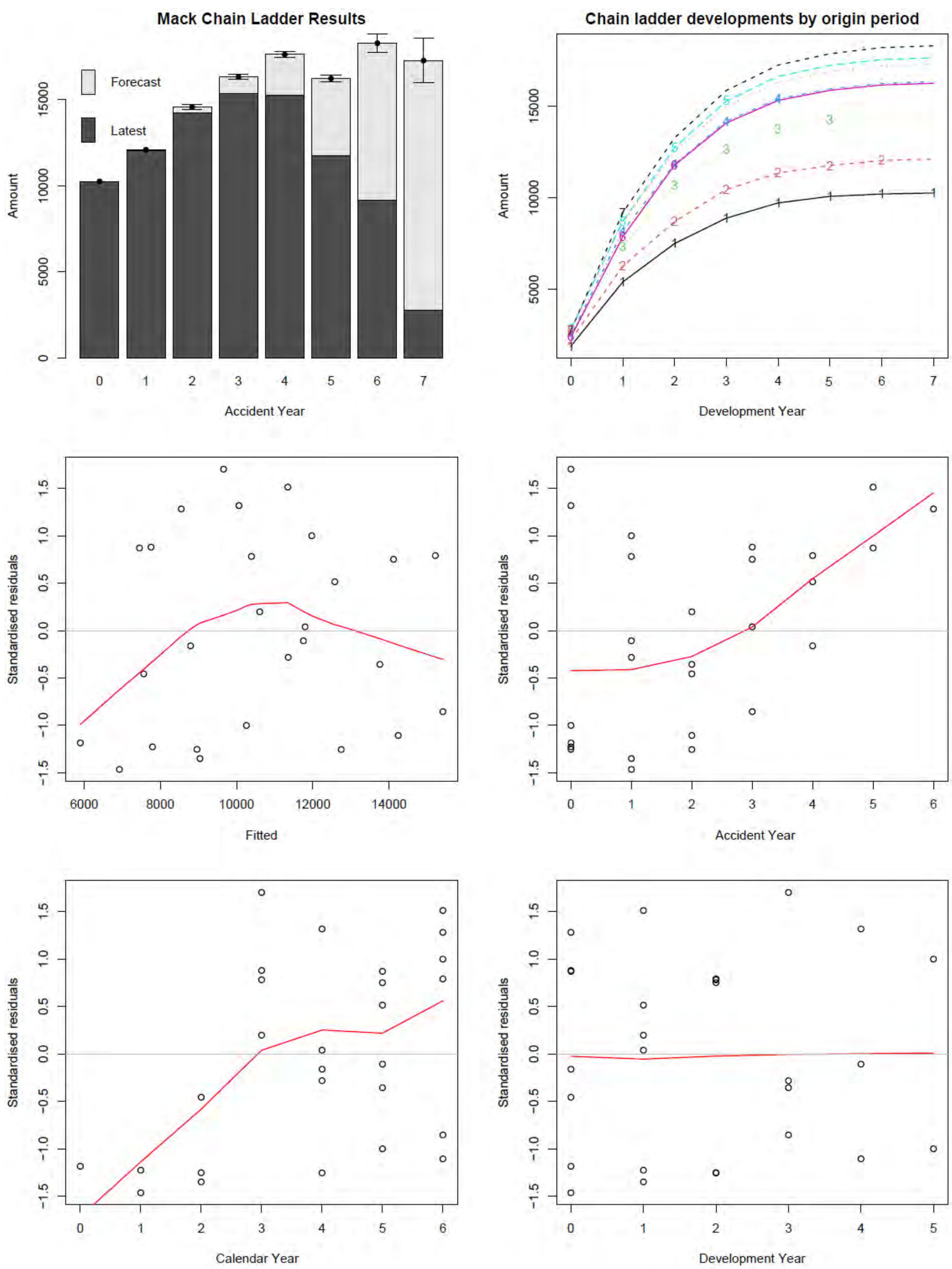
(i) Explain briefly how these assumptions differ from the Chain Ladder assumptions.

(ii) Show that under Mack's model

$$\mathrm{Var}\left[C_{i,j+2} \middle| C_{i,j}\right] = C_{i,j}\left(\sigma_{j+1}^2\ f_j + f_{j+1}^2\ \sigma_j^2\right)$$

6. **Continued**

Your colleague has fitted Mack's model to the runoff triangle in (a), using the R ChainLadder package. The results are summarized in the graphs below. Note that there are 26 residual values shown in each of the lower four residual plots.



R plots for Mack Model fit to Question 6 data

6. Continued

(d) (*2 points*)

(i) State with reasons whether the graphs show that the data are consistent with the Mack Model assumptions.

(ii) Explain briefly how the Mack Model estimates of outstanding claims compare with the chain ladder estimates.

(e) (*2 points*)

(i) You note from the R output that the sum of the Mean Square Errors of Prediction (MSEP) over all the AYs is 1428^2, and that the total MSEP is 1547^2.

Your colleague says that this must be an error, as the model assumption A1 states that each AY is independent. Critique this comment.

(ii) Another colleague notes that the Bornhuetter-Ferguson estimate of the final cumulative claims for AY6 (from (b) above) lies outside the AY6 whiskers in the top left plot. State with reasons which estimate you would recommend to the insurer.

****END OF EXAMINATION****